meetkunde: enkele lemma's

olympia ism QED en VWO

2011

1 basis

In dit bestand wordt de meetkunde van bij de open vragen zo goed mogelijk uitgelegd met enkel de mooie kant van het projectief-synthetisch deel.

De basis bestaat uit:

- isometrieen zoals homotheties, verschuivingen, verdraaiingen
- angle-chasing
- constructie van interessante punten die helpen het probleem op te lossen
- gelijkvormigheid en congruentie
- de eigenschappen van koordenvierhoeken en omtrekshoeken: omtrekshoeken op eenzelfde boog zijn gelijk en dus ook de hoeken op dezelfde zijde binnen een koordenvierhoek, wiens overstaande hoeken een som van 180 graden heeft.
- vectoren.

herh. basiskennis

Een driehoek $\triangle ABC$ is gelijkbenig met $|AB| = |AC|$ als en slechts als de twee hoeken ("basishoeken") B en C gelijk zijn.

In gelijkvormige driehoeken $\triangle ABC$ en$\triangle XYZ$ zijn de overeenkomstige hoeken gelijk en de overeenkomstige zijden hebben een constante verhouding ("de zijden zijn evenredig"). I.e.$A = X, B = Y, C = Z, \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$. Notatie $ABC \sim XYZ$

Twee driehoeken $\triangle ABC$ en $\triangle XYZ$ zijn congruent (gelijkvormig en de overeenkomstige zijden zijn even lang) als een van de volgende voldaan is ("congruentiekenmerken") * Alle overeenkomstige zijden even lang zijn ("ZZZ") * Twee overeenkomstige zijden even lang zijn, en de ingesloten hoeken gelijk zijn ("ZHZ") * Twee overeenkomstige hoeken gelijk zijn, en n overeenkomstig paar zijden even lang is ("HZHën "ZHH")

Voor gelijkvomigheid is het voldoende dat de verhouding van de zijden gelijk is (ZHH wordt dan HH)

Stelling van Pythagoras: in een rechthoekige driehoek met rechte hoek A geldt $|AB|^2 + |AC|^2 = |BC|^2$.

Als twee driehoeken een gemeenschappelijke top hebben, en een basis met dezelfde drager (de drager van een lijnstuk AB is de rechte AB), dan verhouden hun oppervlakten zich als de lengten van hun basissen. Dus: voor driehoeken ABC en ADE, met B, C, D, E colineair (op dezelfde rechte) geldt $[ABC]/[ADE] = BC/DE$. Dit volgt onmiddellijk uit öppervlakte driehoek = basis * hoogte / 2". Hierbij staat $[ABC]$ voor de oppervlakte van de driehoek ABC.

Nog wat naamkennis herhalen:

* De zwaartelijnen van een driehoek (uit een hoekpunt naar het midden van de overstaande zijde) snijden elkaar in n punt, het zwaartepunt Z van de driehoek, in de theorie staat er G

* De hoogtelijnen van een driehoek (uit een hoekpunt loodrecht op de overstaande zijde) snijden elkaar in n punt, het hoogtepunt H van de driehoek.

* De middelloodlijnen van een driehoek (de middelloodlijnen van de zijden) snijden elkaar in n punt, het omcentrum O van de driehoek.

* De bissectrices van een driehoek (de bissectrices van de hoeken) snijden elkaar in n punt, het incentrum I van de driehoek.

* het centrum van de negenpuntscirkel wordt met E aangeduid bij de 3 blz. met theorie, dit is het middelpunt van de cirkel door o.a. de middens van de zijden van de driehoek

eig. Vliegers:

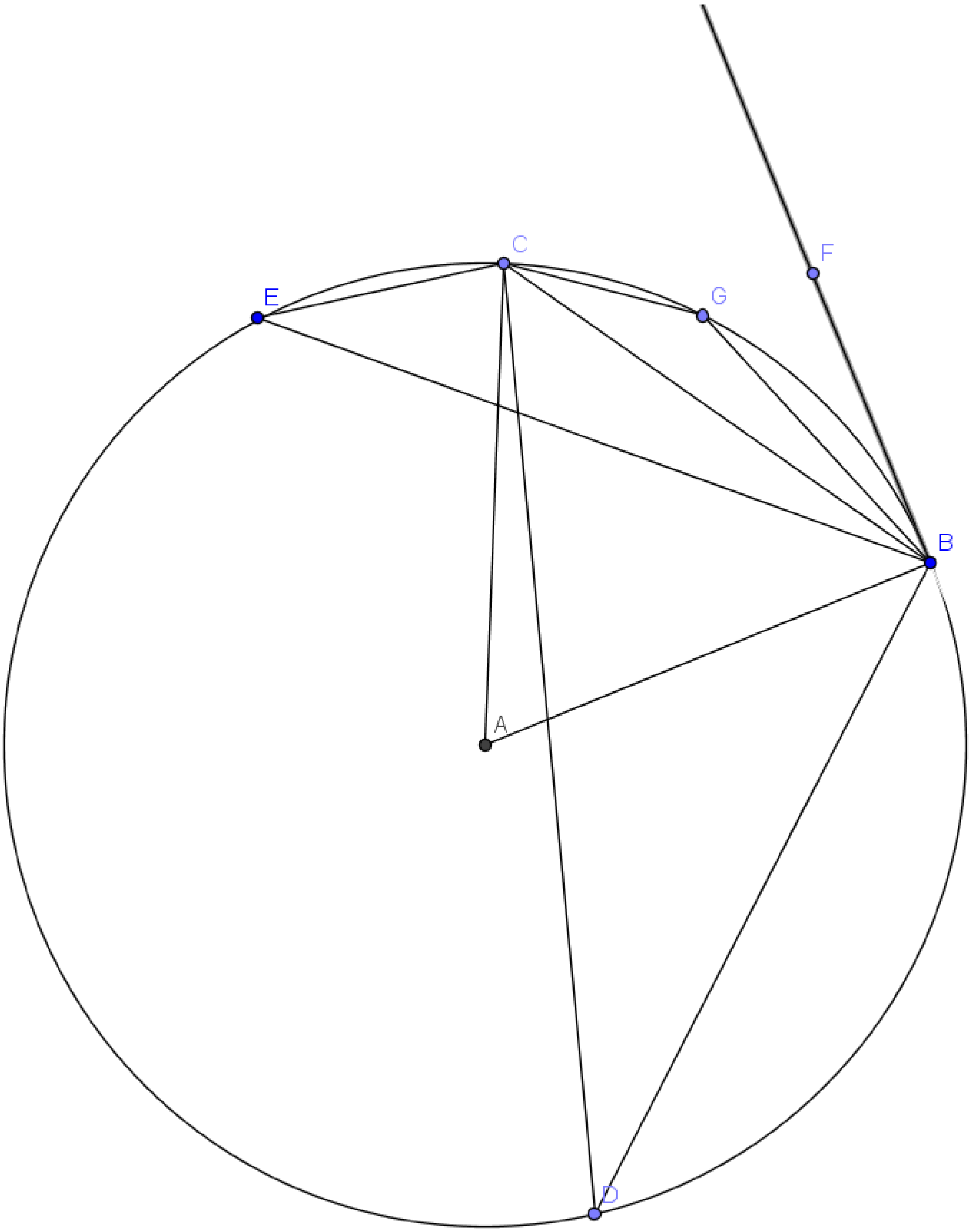
*2 paar aangrenzende zijden zijn even lang en *De diagonalen van een vlieger staan loodrecht op elkaar.

basiseigenschappen over koordenvierhoeken:

*de overstaande hoeken zijn supplementair

*omtrekshoeken op een gelijke boog zijn gelijk, met een waarde die de helft is van de middelpuntshoek op de boog.

Op volgende tekening zien we dus duidelijk dat $\angle CBF = \angle CDB = \angle CEB = 0.5\angle CAB$ met die eigenschappen.



C
E
G
F
B
A
D

De bissectricestelling

Een eigenschap van de bissectrices van een driehoek ABC die in talloze problemen als een zeer fundamentele stelling opduikt is de volgende:

De binnen- en buitenbissectrice van hoek α snijdt BC in D en E respectievelijk. Er geldt dat

$$\frac{AB}{AC} = \frac{BD}{CD} = \frac{BE}{CE}$$

(Bewijs als oefening)

De apoloniuscirkel

Interessant om verder mee te gaan, is de apoloniuscirkel:

Stelling 1.1. *(De cirkel van Apollonius)*

Zij $[AB]$ een lijnstuk en k een positief reel getal ongelijk aan 1. De meetkundige plaats van alle punten P waarvoor geldt $\frac{|PA|}{|PB|} = k$ is een cirkel met middelpunt op de rechte AB.

opmerking: indien $k = 1$ is de meetkundige plaats de middelloodlijn van het lijnstuk

bewijs

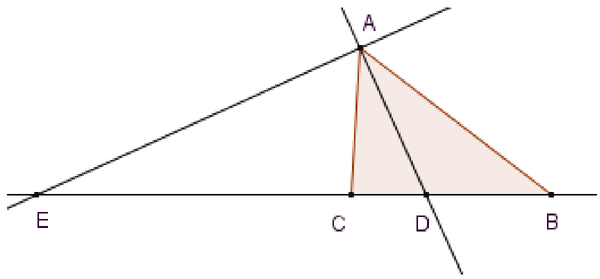
Stel dat we 1 punt Q hebben dat niet op de rechte AB ligt, hebben die voldoet aan de voorwaarde, door de bissectricestelling weten we dat de snijpunten van binnen- en buitenbissectrice van $\triangle AQB$ met de rechte AB ook voldoen aan de voorwaarde. Indien C, D die 2 snijpunten zijn, is de cirkel van Apolonius de cirkel met diameter $[CD]$. Nu kan er ook worden bewezen dat voor ieder punt P op die cirkel de verhouding blijft kloppen. (zij D het punt niet tussen A en B, neem de projecties X, Y op DP en werk met gelijkvormige driehoeken)

Stelling 1.2. *(De vlinderstelling) Laat M het midden zijn van een koorde PQ van een cirkel en AB en CD twee andere koorden door M. Noem X het snijpunt zijn van AD en PQ en Y van BC en PQ. Dan is M het midden van XY.*

Het bewijs is al een uitdaging op zich, de stelling helpt bij vragen zoals bvb.

Voorbeeld 1.3. *BaMO 2008/1*

Gegeven een scherphoekige driehoek ABC met $|AC| > |BC|$ en F als voetpunt van C op $[AB]$. Laat P een punt zijn op AB, $\neq A$ zodat $|AF| = |PF|$. Zij H, O, M het hoogtepunt,omcentrum en midden van $[AC]$. Zij X het snijpunt van BC en HP en Y 't snijpunt van OM en FX, laat OF snijden met AC in Z. Bewijs dat F, M, Y, Z een koordenvierhoek vormen.



Figuur 1: De bissectricestelling

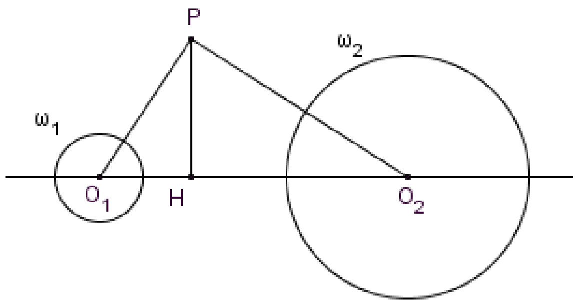
Macht van een punt

Wanneer we een punt P en een cirkel ω met middelpunt O en straal R beschouwen, en we tekenen een willekeurige rechte door P die ω snijdt in A en B, dan merken we op dat de grootte van $|PA|\,.\,|PB|$ onafhankelijk is van de gekozen rechte. (Bewijs dit) We definiëren de *macht van* P *t.o.v.* ω als

- $-\,|PA|\,.\,|PB|$ als P binnen de cirkel ligt.
- 0 als P op de cirkel ligt.
- $|PA|\,.\,|PB|$ als P buiten de cirkel ligt.

Toon nu aan dat de macht van P t.o.v. ω ook gegeven wordt door $|OP|^2 - R^2$. Nu wordt de betekenis van het minteken in de definitie van de macht van P t.o.v. ω duidelijk, ze was nodig om de zonet gegeven uitdrukking steeds te doen kloppen.

Machtlijn



Figuur 2: De machtlijn

Wanneer er twee cirkels in het spel zijn zouden we ons kunnen afvragen wat de *meetkundige plaats* is van alle punten P die t.o.v. beide cirkels dezelfde macht hebben. Beschouw daartoe twee cirkels ω_1, ω_2 met straal r_1, r_2 en middelpunt O_1, O_2. Zij P een punt dat gelijke macht ten opzichte van beide cirkels heeft, en noem H de projectie van P op O_1O_2. Volgende gelijkheden zijn nu equivalent:

$$\begin{aligned}
|O_1P|^2 - r_1^2 &= |O_2P|^2 - r_2^2 \\
|O_1H|^2 + |HP|^2 - r_1^2 &= |O_2H|^2 + |HP|^2 - r_2^2 \\
|O_1H|^2 - r_1^2 &= (|O_2O_1| - |HO_1|)^2 - r_2^2 \\
2 \cdot |HO_1| \cdot |O_2O_1| &= |O_2O_1|^2 + r_1^2 - r_2^2
\end{aligned}$$

Merk op dat de onderste vergelijking enkel afhankelijk is van de positie van H. Het punt P zal met andere woorden dan en slechts dan een gelijke macht hebben t.o.v. beide cirkels als H dat ook heeft. Wanneer $|O_1O_2| \neq 0$ is de laatste vergelijking een eerstegraadsvergelijking die een unieke H oplevert (Merk op dat we met geöriënteerde lengtes werken). Bijgevolg is de meetkundige plaats die we zochten een rechte loodrecht op O_1O_2. Deze rechte wordt ook wel de *machtlijn* van beide cirkels genoemd.

Merk op dat we eenvoudig de machtlijn van twee snijdende cirkels kunnen terugvinden als de rechte door beide snijpunten (of de gemeenschappelijk raaklijn indien de cirkels raken in een punt). Ga na waarom dat zo is.

Machtpunt

Wanneer we drie cirkels beschouwen, dan kunnen we voor elk paar cirkels de machtlijn gaan beschouwen. Bewijs nu zelf de volgende stelling:

Gegeven zijn drie cirkels ω_1, ω_2 en ω_3. De drie machtlijnen die we krijgen door telkens twee verschillende cirkels uit de gegeven drie cirkels te beschouwen zijn concurrent.

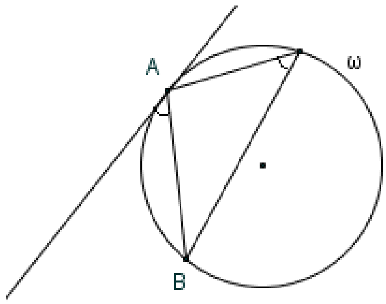
Het punt van concurrentie van deze 3 machtlijnen wordt vaak het *machtpunt* van de drie cirkels genoemd.

Essentiële lemmata

In deze paragraaf bekijken we enkele lemma's van dichterbij die ongewoon vaak hun intrede deden in IMO-problemen de voorbije jaren. Waar Lemma 1 misschien ook als fundamentele stelling bestempeld zou kunnen worden, zijn Lemma 2 en vooral Lemma 3 ongemeen belangrijk voor elke IMO-deelnemer.

Lemma 1 (Raakomtrekshoek) *Beschouw een cirkel ω die de punten A en B bevat. De raaklijn aan ω in A sluit een hoek in met AB die in grootte gelijk is aan een van beide omtrekshoeken op AB in ω.*

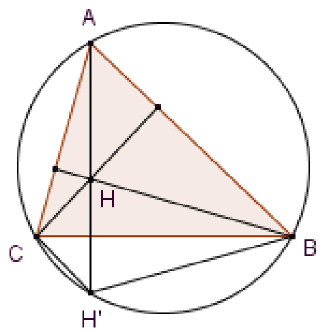
(Bewijs als oefening)



Figuur 3: De raakomtrekshoek

Lemma 2 *De reflecties van het hoogtepunt H van ABC ten opzichte van de zijden liggen op de omgeschreven cirkel van ABC.*

Het bewijs van dit lemma is eenvoudig en kan als oefening dienen voor de lezer.

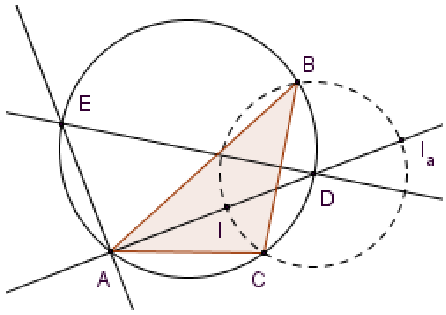


Figuur 4: Lemma 2

Lemma 3 *In driehoek ABC noemen we I het middelpunt van de ingeschreven cirkel, en I_a het middelpunt van de aangeschreven cirkel tegenover A.*

- *De binnen(resp. buiten)bissectrice van A snijdt de middelloodlijn van BC in het punt D (resp. het punt E) op de omgeschreven cirkel.*
- *De cirkel met diameter II_a bevat B en C en heeft D als middelpunt.*

Dit lemma is allicht het belangrijkste uit deze hele paragraaf, en een van de vaakst terugkerende lemmata in oplossingen van IMO-problemen. Het bewijs is een goede oefening.



Figuur 5: Lemma 3

Voetpuntsdriehoeken

Wanneer je in een probleem een punt binnen een driehoek ontmoet, kan het vaak nuttig zijn om de zogenaamde *voetpuntsdriehoek* te beschouwen. De voetpuntsdriehoek van een punt P in een driehoek ABC is per definitie de driehoek gevormd door de loodrechte projecties van P op de zijden van ABC. Zo zullen bijvoorbeeld de middens van de zijden van ABC de hoekpunten van de voetpuntsdriehoek van O zijn. De voetpuntsdriehoek heeft enkele eigenschappen die af en toe tot een zeer korte oplossing van een probleem kunnen leiden.

Eigenschappen *In ΔABC tekent men de voetpuntsdriehoek DEF van P. Er geldt dat*

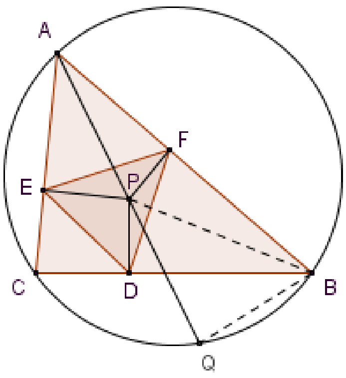
- $\angle EDF = \angle CPB - \angle CAB$
- $|DE| = |CP| \cdot \sin C$
- $\text{Opp(DEF)} = \frac{1}{4}\left|1 - \frac{|OP|^2}{R^2}\right| \cdot \text{Opp(ABC)}.$

Essentieel bestaat het bewijs van de eerste twee eigenschappen uit het optellen van hoeken en het gebruiken van de sinusregel. De derde eigenschap is minder eenvoudig en verdient speciale aandacht. We geven eerst een bewijs:

Noem Q het tweede snijpunt van AP en de omgeschreven cirkel van ABC. We merken op dat $BDPF$ en $AEPF$ koordenvierhoeken zijn, en dus vinden we dat $\angle EFD = \angle EFP + \angle PFD = \angle EAP + \angle PBD = \angle CBQ + \angle PBC = \angle PBQ$. Voor de oppervlakte van DEF vinden we ten slotte:

$$\begin{aligned} 2 \cdot \text{Opp}(DEF) &= |EF| \cdot |DF| \, . \sin EFD \\ &= |AP| \cdot |BP| \cdot \sin A \cdot \sin B \cdot \sin PBQ \\ &= |AP| \cdot |PQ| \cdot \sin A \cdot \sin B \cdot \sin PQB \\ &= |\left(|OP|^2 - R^2\right)| \cdot \sin A \cdot \sin B \cdot \sin C \end{aligned}$$

Hieruit volgt het gestelde.



Figuur 6: De voetpuntsdriehoek

De rechte van Simson

Drie punten zijn collineair als en slechts als de driehoek gevormd door deze drie punten een oppervlakte heeft die nul is. Ga nu zelf met behulp van eigenschap 3 uit de vorige paragraaf de volgende stelling na:

De projecties van een punt P op de zijden van ABC zijn dan en slechts dan collineair als P op de omgeschreven cirkel van ABC ligt.

De rechte die de drie projecties van het punt P bevat noemt men de *rechte van Simson* van punt P t.o.v. ABC.

Deze stelling zal je soms van pas komen wanneer je een probleem te lijf gaat. Tracht ook als oefening eens een rechtstreeks bewijs te vinden, dus zonder de uitdrukking voor de oppervlakte van een voetpuntsdriehoek te gebruiken.

Deze dingen worden uitgelegd in volgende bestand, gevolgd door een vervollediging van de goniometrische eigenschappen

1 Techniques

Geometry is unlike many of the other areas of olympiad mathematics, requiring more intuition and less algebra. Nevertheless, it is important to do the basic groundwork as otherwise your intuition has nothing with which to work.

Here are some suggestions on ways to approach a geometry problem.

- Draw a quick diagram so that you can visualise the problem.
- Draw a neat and accurate diagram — this will often reveal additional facts which you could then try to prove.
- Draw a deliberately incorrect diagram (this could be your initial diagram), so that you don't accidentally assume the result because you referred to your accurate diagram (this is particularly important if you are proving concurrency or collinearity).
- It is very important to do as much investigation as you can. Try to relate as many angles and line segments as you can, even if you have several variables. Then look for similar or congruent triangles, parallel lines and so on. This on its own can be enough to solve some easier problems without even having to think.
- There are many approaches to attack geometry problems e.g. Euclidean geometry, coordinate geometry, complex numbers, vectors and trigonometry. Think about applying all the ones that you know to the problem and deciding which ones are most likely to work. Be guided by what you are asked to prove: for example, if you are asked to prove that two lines are parallel then coordinate geometry might work well, but if the problem involves lots of related angles then trigonometry may be a better approach.
- Don't be afraid to get your hands dirty with trigonometry, coordinate geometry or algebra. While such solutions might not be as "cool" as solutions that require an inspired construction, they are often easier to find and score the same number of points. However, doing as much as possible with Euclidean geometry first can make the equations simpler.
- Look for constructions that will give you similar triangles, special angles or allow you to restate the problem in a simpler way. For example, if you are asked to prove something about the sum of two lengths, try making a construction that places the two lengths end to end so that you only have to prove something about the length of a single line.

- Assume that the result is true, and see what follows from this. This may lead you to intermediate results which you can then try to prove.
- Always check that you haven't omitted any cases such as obtuse angles or constructions that are impossible in certain cases (for example, you can't take the intersection point of two lines if they are parallel). This booklet does a terrible job of this, because the special cases are almost always trivial. I'm lazy, the duplication costs of this booklet are high, the rainforests are dying, and this is not a competition. In a competition, you can expect to lose marks if your proof does not work in all cases.

2 Terminology and notation

There is some basic terminology for things that share some property. Concurrent lines pass through a common point, and collinear points lie on a common line. Concyclic points lie on a common circle; note that "*A*, *B*, *C* and *D* are concyclic" does not have the same meaning as "*ABCD* is a cyclic quadrilateral", since the latter implies that the points lie in a particular order around the circle. Concentric circles have a common centre.

The humble triangle has possibly the richest terminology and notation. There are numerous "centres", generally the point of concurrency of certain lines, and a few have corresponding circles.

incentre The centre of the incircle (inscribed circle); the point of concurrency of the internal angle bisectors

circumcentre The centre of the circumcircle (circumscribed circle); the point of concurrency of the perpendicular bisectors

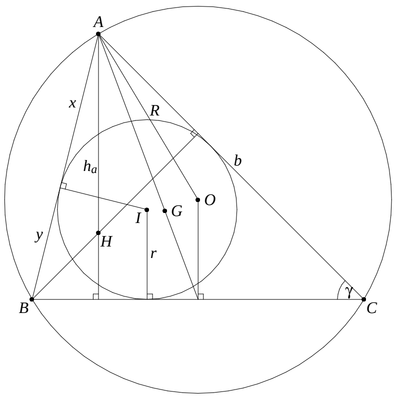
excircle The centre of an excircle (escribed circle); the point of concurrency of two external and one internal angle bisector

orthocentre The point of concurrency of the altitudes

centroid The point of concurrency of the medians (lines from a vertex to the midpoint of the opposite side)

Most of these terms should be familiar from high-school geometry. An unfamiliar term is a *cevian*: this is any line joining a vertex to the opposite side.

For this booklet (particularly section 6), we also introduce a lot of notation for triangles. Some of this is standard or mostly standard while some is not; you are advised to define any of these quantities in proofs, particularly K, x, y and z.



I the incentre

I_A the excentre opposite A

O the circumcentre

G the centroid

H the orthocentre

a the side opposite vertex A (similarly for B and C)

s the semiperimeter, $\frac{a+b+c}{2}$

x the tangent from A to the incircle, $\frac{-a+b+c}{2} = s - a$ (similarly for y and z)

R the radius of the circumcircle (circumradius)

r the radius of the incircle (inradius)

r_a the radius of the excircle opposite A

h_a the height of the altitude from A to BC

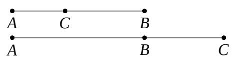
α the angle at A (similarly for β and γ)

K the area of the triangle

We also use the notation $|\triangle ABC|$ (or just $|ABC|$) to indicate the area of $\triangle ABC$.

3 Directed angles, line segments and area

In classical geometry, most quantities are undirected. That means that if you measure them in the opposite direction, they have the same value ($AB = BA$, $\angle ABC = \angle CBA$, and $|\triangle ABC| = |\triangle CBA|$). Most of the time this is a reasonable way of doing things. However, it occasionally has disadvantages. For example, if you know that A, B and C are collinear, and $AB = 5$, $BC = 3$, then what is AC? It could be either 2 or 8, depending on which way round they are on the line. The same problem arises when adding angles or areas.



Normally these situations are not important, because it is clear from a diagram which is correct. However, sometimes there are many different ways to draw the diagram, leading to a proof with many different cases. Another way to solve the problem is to treat the quantities as having a sign, indicating the direction. So now if you are told that $AB = 5$, $BC = 3$ then you can be sure that $AC = AB + BC = 8$. This is because both have the same sign, and hence are in the same direction. If C lay between A and B, then $AB = 5$, $BC = -3$ and so $AC = AB + BC = 2$. It could also be that $AB = -5$, $BC = 3$; the positive direction is generally arbitrary but must be consistent. What is important is that no matter in what order A, B and C lie, the equation $AC = AB + BC$ holds.

Directed line segments have somewhat limited use, because it only makes sense to compare lines that are parallel. Generally they are used when dealing with ratios or products of collinear line segments (see Menelaus' Theorem (6.3), for example). Directed angles and directed area are more often used.

A directed angle $\angle ABC$ is really a measure of the angle between the two lines AB and BC. Conventionally, it is the amount by which AB must be rotated anti-clockwise to line up with BC. One effect of this is that while normal angles have a range of 360°, directed angles only have a range of 180°! This is because rotating a line by 180° leaves it back where it started, so 180° is equivalent to 0°. To indicate this, equivalent angles are sometimes written $\angle ABC \equiv \angle DEF$ rather than $\angle ABC = \angle DEF$. This limitation occasionally has disadvantages, and in particular it is not generally possible to combine trigonometry with directed angles (since the sin and cos functions only repeat every 360°). This is made up for by the special properties that directed angles do have:

1. $\angle AMC \equiv \angle AMB + \angle BMC$;
2. $\angle AXY \equiv \angle AXZ$ iff X, Y, Z are collinear
3. $\angle XYZ \equiv 0°$ iff X, Y, Z are collinear
4. $\angle ABC + \angle BCA + \angle CAB \equiv 0$;
5. $\angle PQS \equiv \angle PRS$ iff P, Q, R and S are concyclic.

Property 1 is simply the basis of directedness: the relative positions don't matter. Property 2 is trivial if Y and Z lie on the same side of X, and the fact that adjacent angles add up to 180° if not. Property 3 just restates the fact that rotating a line onto itself leads to no rotation. Property 4 is the result that angles in a triangle add up to 180°, but also brings in the fact that the three angles are either all clockwise or all anti-clockwise. Property 5 is the really interesting one: it is *simultaneously* the same segment theorem and the alternate segment theorem, depending on the ordering of the points on the circle. The problem below illustrates why having a single theorem can be so important.

Directed areas are used even less often than directed angles and line segments, but are sometimes useful when adding areas to compute the area of a more complex shape. Conventionally, a triangle ABC has positive area if A, B and C are arranged in anti-clockwise order, and negative if they are arranged in clockwise order.

Exercise 3.1. *Three circles, Γ_1, Γ_2 and Γ_3 intersect at a common point O. Γ_1 and Γ_2 intersect again at X, Γ_2 and Γ_3 intersect again at Y, and Γ_3 and Γ_1 intersect against at Z. A is a point on Γ_1 which does not lie on Γ_2 or Γ_3. AX intersects Γ_2 again at B, and BY intersects Γ_3 again at C. Prove that A, Z and C are collinear.*

Exercise 3.2 (Simpson Line). *Perpendiculars are dropped from a point P to the sides of $\triangle ABC$ to meet BC, CA, AB at D, E, F respectively. Show that D, E and F are collinear if and only if P lies on the circumcircle of $\triangle ABC$.*

You will find that directed angles in particular play a large role in the theorems in this book, and they are introduced early on for this purpose. Do not be led to believe that directed angles are so wonderful that they should be used for all problems: theorems try to make very general statements and use directed angles for generality, but most problems are constrained so that normal angles are adequate (e.g. points inside triangles or acute angles). Normal angles are easier to work with simply because one does not need to think about whether to write $\angle ABC$ or $\angle CBA$.

4 Trigonometry

Trigonometry is seldom required to solve a problem. After all, trigonometry is really just a way of reasoning about similar triangles. However, it is a very powerful reasoning tool, and if applied correctly can replace a page full of unlikely and ungainly constructions with a few lines of algebra. If applied incorrectly, however, it can have the opposite effect.

The first thing to do before applying any trigonometry is to reduce the number of variables to the minimum. Then choose the variables that you want to keep very carefully. The compound angle formulae below make it easy to expand out many trig expressions, but if you have chosen the wrong variables to start with the task is almost impossible.

The following angle formulae are invaluable in manipulating trigonometric expressions. In the formulae below, a $\mp$ indicates a sign that is opposite to the sign chosen in a $\pm$.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{4.1}$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{4.2}$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \tag{4.3}$$
$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot A \pm \cot B} \tag{4.4}$$
$$\sin A \sin B = [\cos(A-B) - \cos(A+B)]/2 \tag{4.5}$$
$$\sin A \cos B = [\sin(A-B) + \sin(A+B)]/2 \tag{4.6}$$
$$\cos A \cos B = [\cos(A-B) + \cos(A+B)]/2 \tag{4.7}$$
$$\sin A \pm \sin B = 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right) \tag{4.8}$$
$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \tag{4.9}$$

$$\cos A - \cos B = 2\sin\left(\frac{B+A}{2}\right)\sin\left(\frac{B-A}{2}\right) \tag{4.10}$$

You don't need to memorise any of these other than the first three, because all the others can be obtained from these with simple substitutions. You should be aware that these transformations exist and know how to derive them, so that you can do so in an olympiad if necessary (see the exercises).

You can also use these to derive other formulae; for example, you can calculate $\sin n\theta$ and $\cos n\theta$ in terms of $\sin\theta$ and $\cos\theta$ fairly easily (for small, known values of n).

Exercise 4.1. *Prove equations* (4.4) *to* (4.10)*.*

Exercise 4.2. *In a $\triangle ABC$ (which is not right-angled), prove that*

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

4.1 The extended sine rule

The standard Sine Rule says that

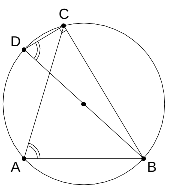
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Theorem 4.1 (Extended Sine Rule). *In a triangle ABC,*

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R is the radius of the circumcircle.

Proof. Construct point D diametrically opposite B in the circumcircle of $\triangle ABC$. Then $\alpha = \angle CDB$ or $180° - \angle CDB$ and $\angle BCD = 90°$. It follows that $\frac{a}{\sin\alpha} = \frac{BC}{BC/BD} = 2R$, and similarly for $\frac{b}{\sin\beta}$ and $\frac{c}{\sin\gamma}$.



□

Exercise 4.3. *In a circle with centre O, AB and CD are diameters. From a point P on the circumference, perpendiculars PQ and PR are dropped onto AB and CD respectively. Prove that the length of QR is independent of the position of P.*

5 Circles

5.1 Cyclic quadrilaterals

A cyclic quadrilateral is a quadrilateral that can be inscribed in a circle. There are several results related to the angles of a cyclic quadrilateral that are covered in high school mathematics and which will not be repeated here. These results are still very important, and cyclic quadrilaterals appear in many unexpected places in olympiad problems.

Exercise 5.1 ($\star$). *Let $\triangle ABC$ have orthocentre H and let P be a point on its circumcircle. Let E be the foot of the altitude BH, let PAQB and PARC be parallelograms, and let AQ meet HR in X.*

(a) Show that H is the orthocentre of $\triangle AQR$.

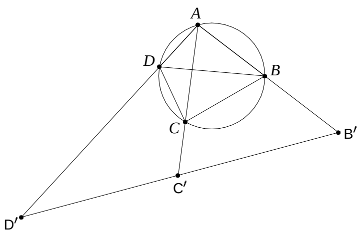
(b) Hence, or otherwise, show that EX is parallel to AP.

A result that is not normally taught in school is Ptolemy's Theorem. It is mainly useful if you have only one or two cyclic quadrilaterals, and lengths play a major role in the problem. It is also very useful when some more is known about the lengths. Equal lengths are particularly helpful as they can divide out of the equation.

Theorem 5.1 (Ptolemy's Theorem). *If ABCD is a cyclic quadrilateral, then*

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

Proof.



Choose an arbitrary constant K and construct B', C' and D' on AB, AC and AD respectively such that $AB \cdot AB' = AC \cdot AC' = AD \cdot AD' = K$.

Now consider $\triangle ABC$ and $\triangle AC'B'$. The angle at A is common and $\frac{AB}{AC'} = \frac{K/AB'}{K/AC} = \frac{AC}{AB'}$ and therefore the triangles are similar. It follows similarly that $\triangle ABD \,|||\, \triangle AD'B'$ and $\triangle ACD \,|||\, \triangle AD'C'$. Hence $\angle B'C'D' = \angle ABC + \angle ADC = 180°$ i.e. $B'C'D'$ is a straight line. From the similar triangles, we have $BC = B'C' \cdot \frac{AB}{AC'} = \frac{B'C' \cdot AB \cdot AC}{K}$, and similarly for CD and BD. Therefore

$$\begin{aligned} AC \cdot BD &= \tfrac{B'D'}{K}(AB \cdot AC \cdot AD) \\ &= (\tfrac{B'C'}{K} + \tfrac{C'D'}{K})(AB \cdot AC \cdot AD) \\ &= AB \cdot CD + AD \cdot BC \end{aligned}$$

This result relies on the fact that $B'C'D'$ is a straight line. If we had used a non-cyclic quadrilateral, this would not have been the case. This shows that the converse of Ptolemy's Theorem is also true. In fact the triangle inequality in $\triangle B'C'D'$ leads to *Ptolemy's Inequality*, which says that $AC \cdot BD \leq AB \cdot CD + AD \cdot BC$ for any quadrilateral $ABCD$, with equality precisely for cyclic quadrilaterals. □

Exercise 5.2. *Triangle ABC is equilateral. For any point P, show that $AP + BP \geq CP$ and determine when equality occurs.*

5.2 The Simpson line

The Simpson line was covered as exercise 3.2, but to emphasise its importance the statement is repeated here. A handy corollary is that the feet of perpendiculars from a point on the circumcircle cannot all meet the sides internally — which can limit the number of cases you need to consider.

Theorem 5.2 (The Simpson line). *Perpendiculars are dropped from a point P to the sides of $\triangle ABC$ to meet BC, CA, AB at D, E, F respectively. Show that D, E and F are collinear if and only if P lies on the circumcircle of $\triangle ABC$.*

This was exercise 3.2, so no proof is provided here.

Exercise 5.3. *From a point E on a median AD of $\triangle ABC$ the perpendicular EF is dropped to BC, and a point P is chosen on EF. Then perpendiculars PM and PN are drawn to the sides AB and AC.*

Now, it is most unlikely that M, E and N will lie in a straight line, but in the event that they do, prove that AP bisects $\angle A$.

5.3 Power of a point

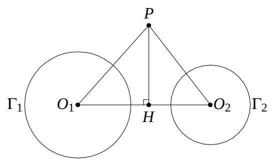
This section is based on the fact that if chords AB and CD of a circle intersect at a point P, then $PA \cdot PB = PC \cdot PD$ (even if P lies outside the circle). This is easily shown using similar triangles.

Consider fixing a point P and circle Γ and considering all possible chords AB that pass through P. Since $PA \cdot PB$ is equal for every pair of chords AB, it is equal for *all* such chords. This value is said to be the power of P with respect to Γ. The line segments are considered to be directed (see section 3), so P is negative inside the circle and positive outside of it. In fact by considering the chord that passes through O, the centre of Γ, it can be seen that the power of P is $d^2 - r^2$, where $d = OP$ and r is the radius of Γ. If P lies outside the circle then this also equals the square of the length of the tangent from P to Γ.

It is sometimes useful to know that the converse of the above result is true i.e. if $PA \cdot PB = PC \cdot PD$, where AB and CD pass through P, then A, B, C and D are concyclic (but only if using directed line segments).

5.3.1 The radical axis

Consider having two circles instead of one. What is the set of points which have the same power with respect to both circles? If the circles are concentric then no point will have the same power (since d will be the same and r different for every point), but the situation is less clear in general.



Consider two circles Γ_1 and Γ_2 with centres O_1 and O_2 with radii r_1 and r_2 respectively. Let P be a point which has equal powers with respect to Γ_1 and Γ_2, and let H be the foot of the perpendicular from P onto O_1O_2. Then

$$
\begin{aligned}
& O_1P^2 - r_1^2 = O_2P^2 - r_2^2 && (5.1)\\
\iff\quad & O_1H^2 + HP^2 - r_1^2 = O_2H^2 + HP^2 - r_2^2 && (5.2)\\
\iff\quad & O_1H^2 - r_1^2 = O_2H^2 - r_2^2 && (5.3)\\
\iff\quad & O_1H^2 - r_1^2 = (O_2O_1 - HO_1)^2 - r_2^2 && (5.4)\\
\iff\quad & 2 \cdot HO_1 \cdot O_2O_1 = O_2O_1{}^2 + r_1^2 - r_2^2 && (5.5)
\end{aligned}
$$

We have eliminated P from the equation! In fact (5.3) shows that P has equal powers with respect to the circles iff H does. If $O_1O_2 \neq 0$ then we have a linear equation in HO_1 and so there is exactly one possibility for H (we are using directed line segments, so HO_1 uniquely determines H). Thus the locus of P is the line through H perpendicular to O_1O_2. This line is known as the *radical axis* of Γ_1 and Γ_2.

If the two circles intersect, the radical axis is easy to construct. The points of intersection both have zero power with respect to both circles, so both points lie on the radical axis. So the radical axis is simply the line through them.

Exercise 5.4. *Two circles are given. They do not intersect and neither lies inside the other. Show that the midpoints of the four common tangents are collinear.*

5.3.2 Radical centre

What happens when we consider three circles (say Γ_1, Γ_2 and Γ_3) instead of two? Firstly consider the case where the centres are not collinear. Then the radical axis of Γ_1 and Γ_2 will meet the radical axis of Γ_2 and Γ_3 at some point, say X (they will not be parallel because a radical axis is perpendicular to the line between the centres of the circles). Then from the definition of a radical axis, X has the same power with respect to all three circles and so it also lies on the radical axis of Γ_1 and Γ_2. The fact

that the three radical axes are concurrent at a point (known as the *radical centre*) can be used to solve concurrency problems.

If, however, the three centres are collinear, then all three radical axes are parallel. If they all coincide then all points on the common axis have equals powers with respect to the three circles; if not then no points do.

Exercise 5.5. *Show how to construct, using ruler and compass, the radical axis of two non-intersecting circles.*

Exercise 5.6 ($\star$). *Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.*

6 Triangles

6.1 Introduction

A triangle would seem to be almost the simplest possible object in geometry, second only to the circle. It has only has two true degrees of freedom, since scaling a triangle up or down does not affect its properties. Yet the humble triangle contains an enormous amount of mathematics — in fact too much to fully explore here.

6.2 Tangents to the incircle

Let the lengths of the tangents to the incircle from A, B and C be x, y and z. Since $a = y+z$, $b = z+x$ and $c = x+y$, we can solve for x, y and z and get

$$x = \frac{-a+b+c}{2}, \quad y = \frac{a-b+c}{2}, \quad z = \frac{a+b-c}{2}.$$

This is the same notation that is introducted in section 2.

Exercise 6.1. *Determine the lengths of the tangents from B and C to the excircle opposite A.*

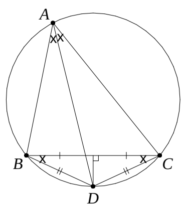
6.3 Triangles within triangles

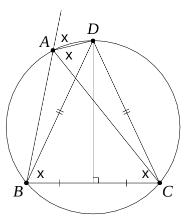
There are specific names given to certain triangles formed from points of the original triangle:

- The *medial* triangle has the midpoints of the original sides as its vertices.
- The *orthic* triangle has the feet of the altitudes as its vertices.
- A *pedal* triangle is the triangle formed by the feet of perpendiculars dropped from some point onto the three sides. If the point is the orthocentre, then this is the orthic triangle (and in fact some people use the term "pedal triangle" to refer to the orthic triangle).

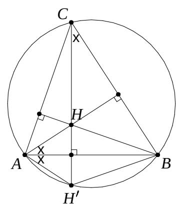
6.4 Points on the circumcircle

Apart from the vertices, there are a few other points that are known to lie on the circumcircle. The first is the intersection point of a perpendicular bisector and the corresponding angle bisector. This is easily shown by taking the intersection of the perpendicular bisector and the circumcircle, which divides an arc (say BC) into two equal parts which subtend equal angles at A. This is also true (although less well known) in the case where the *external* angle bisector is used.





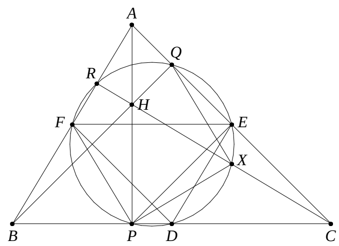
The second group of points that are known to lie on the circumcircle are the reflections of H (the orthocentre) in each of the three sides. This is an exercise in angle chasing, using the known results about the angles in cyclic quadrilaterals.



Exercise 6.2. *A rectangle HOMF has $HO = 23$ and $OM = 7$. Triangle ABC has orthocentre H and circumcentre O. The midpoint of BC is M and F is the foot of the altitude from A. Determine the length of side BC.*

6.5 The nine-point circle

A rather interesting circle that arises in a triangle is the so-called nine-point circle. Let us examine the circumcircle of the triangle whose vertices are the midpoints of $\triangle ABC$ (the medial triangle). Firstly, what is its radius? The medial triangle is a half sized version of the original triangle (because of the midpoint theorem), so its circumradius will also be half that of the large triangle, i.e. it will be $\frac{R}{2}$.



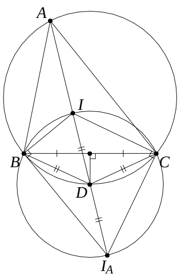
Now let us see what other points this circle passes through. From the diagram it appears that it passes through the feet of the altitudes, so let us prove this. Since F is the midpoint of the hypotenuse of $\triangle APB$, we have $\angle FPA = \angle FAP = 90° - \beta$. Similarly $\angle EPA = 90° - \gamma$ and so $\angle FPE = \alpha = \angle FDE$ (since $\triangle ABC \,|||\, \triangle DEF$). It follows that P lies on the circle. Similarly Q and R also lie on the circle.

Point X is the midpoint of HC, and it also appears to lie on the circle. HC is the diameter of the circle passing through H, Q, C and P, so X is the centre of this circle. It follows that $\angle PXQ = 2\angle PCQ = 2\gamma$. But $\angle PEQ = \angle PEF + \angle FEQ = \angle PDF + \angle FEQ = \gamma + \gamma$, so $\angle PEQ = \angle PXQ$ and so X lies on the circle. Similarly the midpoints of HA and HC lie on the circle.

Because there are nine well-defined points which lie on this circle, it is known as the nine-point circle.

6.6 Another circle

Consider that $\angle I_ABI = \angle I_ACI = 90°$; this shows that II_A is the diameter of a circle passing through I, I_A, B and C. Where is the centre of this circle? Well, any circle passing through B and C must have its centre on the perpendicular bisector of BC, and for II_A to be the diameter, the centre must also lie on the internal bisector of $\angle A$. Hence the centre is the intersection of these two lines. As shown above, the intersection also lies on the circumcircle of $\triangle ABC$.



Exercise 6.3 ($\star$). *In acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that*

(i) the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$;

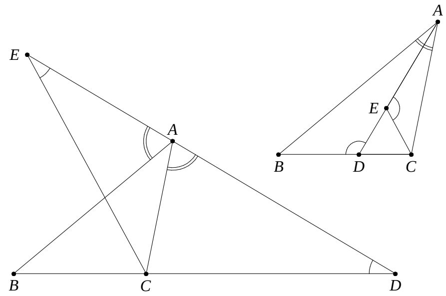
(ii) the area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

6.7 Theorems

Angle bisectors can be fairly tricky to deal with. The angle bisector theorem provides a way to compute the segments which the base is divided into.

Theorem 6.1 (Angle bisector theorem). *If D is the point of intersection of BC with an angle bisector of $\angle A$, then $\frac{DB}{DC} = \frac{AB}{AC}$.*

Proof. Construct E on AD such that $\angle AEC = \angle BDA$. Then $\triangle ABD \,|||\, \triangle ACE$ (two angles) and so $\frac{DB}{EC} = \frac{AB}{AC}$. But $\triangle ECD$ is isosceles, so $CE = CD$ and therefore $\frac{DB}{DC} = \frac{AB}{AC}$ as required.



□

Exercise 6.4. *In the right-hand diagram for the angle-bisector theorem, find a formula for the length BD in terms of the side lengths a, b and c.*

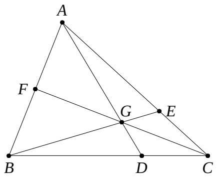
Exercise 6.5. *Given a line segment AB and a real number $r > 0$, find the locus of points P such that $\frac{AP}{BP} = r$.*

The theorems of Ceva and Menelaus are handy results when proving concurrency and collinearity respectively. They are particularly powerful because their converses are true, provided that the directions are taken into account. The converses are quite easy to prove by assuming them to be false, and then constructing two different points with the same uniquely defining properties.

Theorem 6.2 (Ceva's Theorem). *If AD, BE and CF are concurrent cevians of $\triangle ABC$ then*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Proof.



Let G be the point of concurrency.

$$\begin{aligned} \frac{|\triangle ABD|}{|\triangle ACD|} &= \frac{BD}{DC} \quad \text{(common height)} \\ \frac{|\triangle GBD|}{|\triangle GCD|} &= \frac{BD}{DC} \quad \text{(common height)} \\ \therefore \quad \frac{|\triangle AGB|}{|\triangle CGA|} &= \frac{BD}{DC} \end{aligned}$$

We can show similar things for $\frac{CE}{EA}$ and $\frac{AF}{FC}$. Therefore

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{|\triangle AGB|}{|\triangle CGA|} \cdot \frac{|\triangle BGC|}{|\triangle AGB|} \cdot \frac{|\triangle CGA|}{|\triangle BGC|} = 1$$

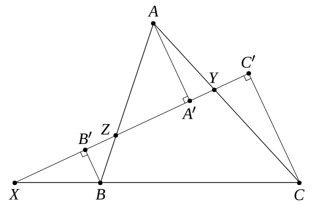
This proof has not explicitly invoked directed areas or line-segments, but if they are used it can be seen that the result will hold even if G lies outside of the triangle. □

Theorem 6.3 (Menelaus' Theorem). *If X, Y and Z and collinear and lie on sides BC, CA and AB (or their extensions) of $\triangle ABC$ respectively, then*

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1$$

(Note that the sign on the result is due to directed line segments, and indicates that the line cuts the sides themselves either twice or not at all.

Proof.



Drop perpendiculars from A, B and C to meet XYZ at A', B' and C'. From alternate angles, we have $\triangle AA'Z \,|||\, \triangle BB'Z$ and thus $\frac{AZ}{ZB} = \frac{AA'}{B'B}$. Similarly $\frac{BX}{XC} = \frac{BB'}{C'C}$ and $\frac{CY}{YA} = \frac{CC'}{A'A}$. Therefore

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{AA'}{B'B} \cdot \frac{BB'}{C'C} \cdot \frac{CC'}{A'A} = -1$$

□

Exercise 6.6. *Use Menelaus' Theorem to prove Ceva's Theorem.*

Exercise 6.7 ($\star$). *ABC is an isosceles triangle with $AB = AC$. Suppose that*

(i) M is the midpoint of BC and O is the point on the line AM such that $OB \perp AB$;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F lies on the line AC such that E, Q and F are distinct and collinear.

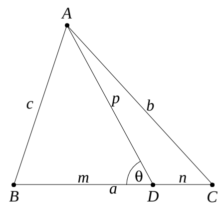
Prove that OQ is perpendicular to EF if and only if $QE = QF$.

Stewart's Theorem is a handy tool for dealing with the length of a cevian, which is otherwise difficult to work with.

Theorem 6.4 (Stewart's Theorem). *Suppose AD is a cevian in $\triangle ABC$. Let $p = AD$, $m = BD$ and $n = CD$. Then*

$$a(p^2 + mn) = b^2 m + c^2 n.$$

Proof.



Use the cosine rule in $\triangle ABD$:

$$c^2 = m^2 + p^2 - 2mp\cos\theta$$
$$\therefore \quad c^2 n = m^2 n + p^2 n - 2mnp\cos\theta \tag{6.1}$$

Do the same in $\triangle ACD$, noting that $\cos(180^\circ - \theta) = -\cos\theta$:

$$b^2 = n^2 + p^2 + 2np\cos\theta$$
$$\therefore \quad b^2 m = n^2 m + p^2 m + 2mnp\cos\theta \tag{6.2}$$

Now add (6.1) and (6.2):

$$b^2 m + c^2 n = m^2 n + n^2 m + p^2 n + p^2 m \tag{6.3}$$
$$= (m+n)(p^2 + mn) \tag{6.4}$$
$$= a(p^2 + mn) \tag{6.5}$$

□

In the special case that AD is a median, Stewart's Theorem reduces to $4p^2 + a^2 = 2(b^2 + c^2)$, which is known as Apollonius' Theorem.

Exercise 6.8. *In $\triangle ABC$, angle A is twice angle B. Prove that $a^2 = b(b+c)$.*

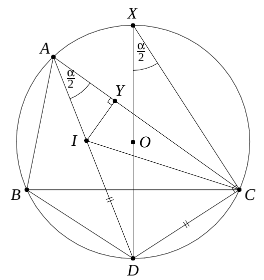
Theorem 6.5 (Euler's Formula).

$$OI^2 = R(R - 2r)$$

As a corollary, we have Euler's Inequality:

$$R \geq 2r.$$

Proof. Extend the angle bisector from A to meet the circumcircle again at D. Also construct X diametrically opposite D on the circumcircle and construct Y as the foot of the perpendicular from I onto AC. We calculate the power of I with respect to the circumcircle (see section 5.3), which is equal to $OI^2 - R^2$ and also to $-AI \cdot ID$. From section 6.6, we have $ID = CD$.



Now we note that $\triangle DXC \,|||\, \triangle IAY$, and so $\frac{AI}{IY} = \frac{XD}{DC} \iff AI \cdot ID = 2rR$. Since $OI^2 - R^2 = -AI \cdot ID$, it follows that $OI^2 = R(R - 2r)$ as required. □

Euler's Theorem provides a measure of the distance between the incentre and circumcentre. However it is most often invoked as Euler's Inequality.

Exercise 6.9 (⋆). *Let r be the inradius and R the circumradius of ABC and let p be the inradius of the orthic triangle of triangle ABC. Prove that*

$$\frac{p}{R} \leq 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2.$$

6.8 Area

There are numerous formulae for the area of a triangle, and in many cases things can be discovered by equating them.

Theorem 6.6 (Heron's Formula).

$$K = \sqrt{sxyz}$$

Proof. This is probably the ugliest proof in this booklet. Here goes:

$$\begin{aligned}
16K^2 &= 4(ab\sin\gamma)^2 \\
&= 4a^2b^2(1-\cos^2\gamma) \\
&= 4a^2b^2\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] \\
&= 4a^2b^2-(a^2+b^2-c^2)^2 \\
&= (2ab-a^2-b^2+c^2)(2ab+a^2+b^2-c^2) \\
&= \left[c^2-(a-b)^2\right]\left[(a+b)^2-c^2\right] \\
&= (c-a+b)(c+a-b)(a+b+c)(a+b-c) \\
&= 16sxyz.
\end{aligned}$$

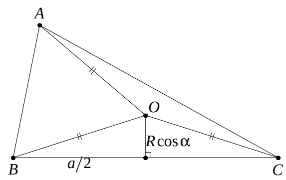
□

Theorem 6.7 (Triangle area formulae).

$$\begin{aligned}
K &= \tfrac{1}{2}ah_a = \tfrac{1}{2}bh_b = \tfrac{1}{2}ch_c && (6.6)\\
&= \tfrac{1}{2}ab\sin\gamma = \tfrac{1}{2}bc\sin\alpha = \tfrac{1}{2}ca\sin\beta && (6.7)\\
&= \frac{abc}{4R} && (6.8)\\
&= 2R^2\sin\alpha\sin\beta\sin\gamma && (6.9)\\
&= \tfrac{1}{2}R(a\cos\alpha+b\cos\beta+c\cos\gamma) && (6.10)\\
&= R(a\cos\beta\cos\gamma+b\cos\gamma\cos\alpha+c\cos\alpha\cos\beta) && (6.11)\\
&= rs && (6.12)\\
&= r_ax = r_by = r_cz && (6.13)\\
&= \sqrt{sxyz} \qquad \textit{(Heron's Formula)} && (6.14)
\end{aligned}$$

Proof. The first is the standard formula for the area of a triangle. The second is really the same formula, since $\sin\gamma = \frac{h_a}{b}$. The third is obtained using the extended sine rule ($\sin\gamma = \frac{c}{2R}$). The fourth is similarly obtained using the extended sine rule by converting all side lengths to sines.

Equation 6.9 is obtained by adding the areas of the isosceles triangles $\triangle BOC$, $\triangle COA$ and $\triangle AOB$. The base of $\triangle BOC$ is a and $\angle BOC = 2\angle BAC = 2\alpha$, so the height is $OC\cos\alpha = R\cos\alpha$. Adding up the areas gives the result.



The following equation is obtained from 6.9 by replacing a by $b\cos\gamma + c\cos\beta$ and similarly for b and c.

Equation 6.12 is obtained similarly to 6.9, but using I instead of O. The three triangles all have height r, so the area is $\frac{1}{2}(ra+rb+rc) = rs$. Equation 6.13 uses the excentre I_a instead; in this case one adds triangles ABI_a and ACI_a and subtracts triangle BCI_a.

Heron's Formula was covered earlier. □

Exercise 6.10. *An equilateral triangle has sides of length $4\sqrt{3}$. A point Q is located inside the triangle so that its perpendicular distances from two sides of the triangle are* 1 *and* 2*. What is the perpendicular distance to the third side?*

Exercise 6.11. *Prove that*

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

There is one area more formula that is used with coordinate geometry.

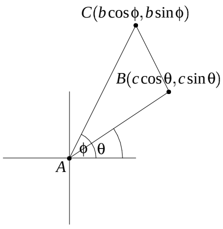
Theorem 6.8. *If one vertex of a triangle is at the origin and the other two are at (x_1,y_1) and (x_2,y_2), then*

$$K = \tfrac{1}{2}\left|x_1y_2 - x_2y_1\right|.$$

If the absolute value operator is removed, one gets a formula for directed area[1].

Proof. The proof below uses trigonometry. It is also possible to compute the area of the triangle by starting with a rectangle that bounds it, and subtracting right triangles. However, that approach requires several cases to be considered.

[1] The sign is used in computer graphics to determine whether three points are wound clockwise or anti-clockwise.



Assume without loss of generality that C makes a larger angle from the x-axis than B (swapping B and C simply negates the term inside the absolute value). Then $(x_1, y_1) = (c\cos\theta, c\sin\theta)$, $(x_2, y_2) = (b\cos\phi, b\sin\phi)$ and the area is

$$\begin{aligned}\tfrac{1}{2}bc\sin\alpha &= \tfrac{1}{2}bc\sin(\phi-\theta)\\ &= \tfrac{1}{2}bc(\sin\phi\cos\theta-\cos\theta\sin\phi\\ &= \tfrac{1}{2}(x_1y_2-x_2y_1).\end{aligned}$$

□

6.9 Inequalities

Inequalities in triangles are often best solved by first expressing all the quantities in terms of as few variables as possible (ideally, only two or three) and then using inequality techniques discussed in *Inequalities for the Olympiad Enthusiast* to finish the problem algebraically. Jensen's Inequality is particularly powerful when combined with trigonometric functions.

Theorem 6.9 (Jensen's Inequality). *A function f is said to be* convex *on an interval $[a,b]$ if $\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$ for all $x, y \in [a,b]$. If f is convex*[2] *on $[a,b]$ then for any $x_1, x_2, \ldots, x_n$ in $[a,b]$ we have*

$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \leq \frac{f(x_1)+\cdots+f(x_n)}{n}.$$

The statement also holds if all inequality signs are reversed, in which case the function is termed concave.

[2] If you are familiar with calculus, a convex function is one that satisfies $f''(x) \geq 0$ for all $x \in [a,b]$.

Proof. Refer to page 18 of *Inequalities for the Olympiad Enthusiast*, by Graeme West. □

Exercise 6.12. *If α, β, γ are the angles of a triangle, then show that $\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\sqrt{3}}{2}$.*

One thing to keep in mind is the triangle inequality: if you reduce the problem to an inequality in a, b and c then it is possible (although not necessarily the case) that you will need to use the fact that the sum of any two is greater than the third. A technique that sometimes simplifies this to substituting $a = x+y$, $b = y+z$, $c = z+x$ in which case the triangle inequality is equivalent to $x, y, z > 0$. In some circles this has become known as the Ravi Substitution, after a Canadian IMO contestant (and later coach) Ravi Vakil. Although he did not invent the technique, he successfully applied it to an IMO problem.

There are a few other useful inequalities that are specific to triangles. The first is Euler's Inequality, mentioned above. The others are listed below.

Theorem 6.10. *In a triangle ABC,*

$$\frac{3\sqrt{3}}{2}R \geq s \qquad s^2 \geq 3\sqrt{3}K \qquad K \geq 3\sqrt{3}r^2.$$

In each case, equality occurs iff $\triangle ABC$ is equilateral.

Proof. We first prove that $\frac{3\sqrt{3}}{2}R \geq s$. From the extended sine rule, $\frac{a}{2R} = \sin\alpha$ and so

$$\begin{aligned}\frac{s}{R} &= \sin\alpha+\sin\beta+\sin\gamma\\ &\leq 3\sin\left(\frac{\alpha+\beta+\gamma}{3}\right) \qquad \text{(Jensen's Inequality)}\\ &= 3\sin 60^\circ\\ &= \frac{3\sqrt{3}}{2}.\end{aligned}$$

For the remaining inequalities, we express everything in terms of x, y and z. Thus

$$\begin{aligned}s^2 &= s^{3/2}\sqrt{s}\\ &= \sqrt{s(x+y+z)^3}\\ &\geq \sqrt{27sxyz} \qquad \text{(AM-GM)}\\ &= 3\sqrt{3}K \qquad \text{(Heron's Formula)}.\end{aligned}$$

$$
\begin{aligned}
K &= \frac{r^2 s^2}{K} \\
&\geq \frac{3\sqrt{3}r^2 K}{K} \qquad \text{(from the previous step)} \\
&= 3\sqrt{3}r^2.
\end{aligned}
$$

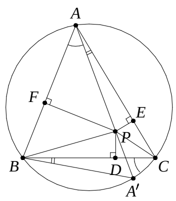
□

Theorem 6.11 (Erdős-Mordell). *Let P be a point inside triangle $\triangle ABC$, and let the feet of the perpendiculars from P to BC, CA, AB be D, E, F respectively. Then*

$$AP + BP + CP \geq 2(DP + EP + FP).$$

Proof. Extend AP to meet the circumcircle of $\triangle ABC$ at A'. Let $\angle BAP = \theta$ and $\angle CAP = \phi$. Note that $FP = AP\sin\theta$ and $EP = AP\sin\phi$, so $\frac{EP}{FP} = \frac{\sin\phi}{\sin\theta} = \frac{CA'}{BA'}$. Also note that $a \cdot AA' = b \cdot BA' + c \cdot CA'$ (from Ptolemy's Theorem in the cyclic quadrilateral $ACA'B$), so $AA' = \frac{b}{a} \cdot BA' + \frac{c}{a} \cdot CA'$. Now

$$
\begin{aligned}
AP &= \frac{FP}{\sin\theta} \\
&= \frac{FP \cdot 2R}{BA'} \qquad \text{(Extended Sine Rule)} \\
&\geq \frac{FP \cdot AA'}{BA'} \qquad \text{(AA' is less than the diameter)} \\
&= \frac{FP(b \cdot BA' + c \cdot CA')}{a \cdot BA'} \\
&= \frac{b}{a} \cdot FP + \frac{c}{a} \cdot \frac{CA'}{BA'} \cdot FP \\
&= \frac{b}{a} \cdot FP + \frac{c}{a} \cdot EP.
\end{aligned}
$$



Now we can establish similar inequalities for BP and CP, and adding these gives

$$
\begin{aligned}
PA + PB + PC &\geq \left(\frac{b}{c} + \frac{c}{b}\right) PD + \left(\frac{c}{a} + \frac{a}{c}\right) PE + \left(\frac{a}{b} + \frac{b}{a}\right) PF \\
&\geq 2(PD + PE + PF). \qquad \text{(AM-GM)}
\end{aligned}
$$

□

Exercise 6.13. *Let ABC be a triangle and P be an interior point in ABC. Show that at least one of the angles PAB, PBC, PCA is less than or equal to 30 degrees.*

7 Transformations

A very powerful idea in geometry is that of a transformation. A transformation maps every point in space to some other point in space. Structures like lines or circles are transformed by applying the transformation to every point on them. They do not necessary maintain their shapes; in fact there is a transformation (inversion) which generally maps lines to circles! Each transformation will preserve certain properties of a diagram, and by translating the properties of the original into the transformed diagram one can obtain new information. Here a diagram is really just a set of points.

7.1 Affine transformations

The transformations we discuss here are all *affine*. That means that straight lines are mapped to straight lines, and lengths are scaled uniformly. The transformations presented here all preserve angles as well. These transformations can in fact be built up by combining reflections and scale changes, although this is not necessarily the best way to think about them.

7.2 Translations, rotations and reflections

The simplest transformation is a translation: every point simply moves a constant distance in a constant direction; this is like picking up a piece of paper and moving it, without rotating it. Rotations rotate all the points by some angle around a particular point; this is like sticking a pin in a piece of paper and then turning it. Reflections take all points and reflect them in a particular line; this is like picking up the piece of paper and putting it down upside-down (the paper would of course need to be thin enough for the diagram to be seen through the back).

While these are all quite straightforward, they can also be very powerful because they preserve so much. They are also closely related, as shown by the next problem.

Exercise 7.1. *In each of the following, show that the transformations exist using a concrete construction.*

(a) Show that any rotation or translation can be expressed as the combination of a pair of reflections, or vice versa.

(b) Show that two rotations, two translations or a translation and rotation can always be combined to produce a single translation or rotation.

(c) Show that any combination of translations, reflections and rotations yields either a rotation, a translation, or a translation followed by a reflection.

Exercise 7.2. *In acute-angled triangle ABC, a point P is given on side BC. Show how to find Q on CA and R on AB such that* $\triangle PQR$ *has the minimum perimeter.*

Exercise 7.3 (⋆). *The point O is situated inside the parallelogram ABCD so that* $\angle AOB + \angle COD = 180^\circ$. *Prove that* $\angle OBC = \angle ODC$.

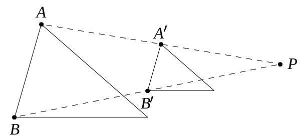
7.3 Homothetisms

So far we have discussed only *rigid* transforms, namely those that can be illustrated with a piece of paper. We now move on to scaling. Imagine drawing a diagram on a new T-shirt, and then letting the T-shirt shrink in the wash. Assume the ink doesn't run and that the T-shirt doesn't warp, you will have the same diagram, only smaller. All the angles and so on will be the same, although lengths will not.

A *homothetism* is a fancy name for scaling. One chooses a centre (sometimes called the "centre of similitude") and a scale factor. Every point is then kept in the same direction relative to the centre, but its distance from the centre is scaled by the scale factor. Like translations, homothetisms preserve orientation, angles, and ratios of lengths. However, lengths are scaled by the scale factor. The result below allows one to find the centre of a homothetism.

Theorem 7.1. *Let S and T be two similar figures which have the same orientation, but are not the same size. Then there is a homothetism that maps S to T.*

Proof. Pick a point A in S and its corresponding point A' in T. Now pick a second point B in S, not on AA', and its corresponding point in B[3]. Now if AA' and BB' are parallel then $AA'B'B$ would be a parallelogram, making $AB = A'B'$. But we assumed that S and T are of different sizes, which would give a contradiction. Hence AA' and BB' meet at a point, which we will call P. Now consider the homothetism with centre of similitude P and scale factor $\frac{A'P}{AP}$. It will clearly map A to A'; will it map B to B'? Yes, because $\triangle ABP \,|||\, \triangle A'B'P$ by parallel lines. If we can show that this homothetism maps the rest of S to T then we are done.



Let C be some arbitrary point in S. We aim to show that the homothetism maps C to its corresponding point C' in T. If C is A or B then we are done. If C lies on AB then C is uniquely defined by $\frac{AC}{BC}$ (with directed line segments). But homothetisms preserve ratios of lengths, and $\frac{A'C'}{B'C'} = \frac{AC}{BC}$ so C is mapped to C'. If C does not lie on AB then C is uniquely defined by the directed angles $\angle BAC$ and $\angle ABC$, and angles are preserved by homothetisms. □

The construction also suggests how the centre of similitude can be found in practice: take two pairs of corresponding points and find the intersection of the lines between them. For example, any two circles of different sizes satisfy the requirements, so a homothetism can be found between them. The points of tangency of the common tangent are corresponding points, since they have the same orientation relative to the centre. Hence the centre of similitude is the intersection of the common tangents.

What happens if we have non-overlapping circles, and use the *other* pair of common tangents? It turns out that this point is also a centre of similitude. However, this homothetism has a negative scale factor, which means that points are "sucked" through the centre and pushed out the other side. This also rotates the figure by 180°, although for a circle this isn't visible. The theorem above in fact applies to situations where the two figures have orientations that are out by 180°, in which case a negative scale factor is used. In this case the figures may even by the same size (since the scale factor is -1, not 1).

Exercise 7.4. *Let ABC be a triangle. Use a homothetism to show that*

(a) the medians of $\triangle ABC$ *are concurrent;*

[3] If no such B exists, then make some arbitrary construction in S and the corresponding construction in T to produce such a B.

(b) *the point of concurrency (the centroid) divides the medians in a* $2:1$ *ratio;*

(c) *the orthocentre* H*, the centroid* G *and the circumcentre* O *are collinear, with* $HG:GO = 2:1$ *(this line is known as the Euler line). Assume that* H *and* O *exist (i.e. that the defining lines are concurrent).*

Exercise 7.5 (⋆). *On a plane let* C *be a circle,* L *be a line tangent to the circle* C *and* M *be a point on* L*. Find the locus of all points* P *with the following property: there exist two points* Q, R *on* L *such that* M *is the midpoint of* QR *and* C *is the inscribed circle of triangle* PQR*.*

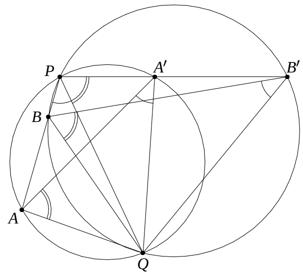
7.4 Spiral similarities

An even more general transformation than a homothetism is a spiral similarity. A spiral similarity combines the effects of a homothetism and a rotation: the plane is not only scaled around a centre P by some factor r, it is also rotated around P by an angle θ. A spiral similarity preserves pretty much the same things as homothetisms i.e. ratio of lengths and angles. However, corresponding lines are no longer parallel, but meet each other at an angle of θ. As for homothetisms, there is a result that makes it possible to find a spiral similarity given two similar figures.

Theorem 7.2. *Let* S *and* T *be two sets of points that are similar but have either different orientation or different size (or both). Then there is a spiral similarity that maps* S *to* T*.*

Proof. In the special case that S and T have the same orientation, there exists a homothetism, which is just a special case of a spiral similarity. So we assume that S and T have different orientations. We also include the case where S and T are oriented 180° apart in the special case, as this is a homothetism with negative scale factor.

Choose two arbitrary points A and B in S, and their corresponding points A' and B' in T. Let P be the intersection of AB and $A'B'$. Construct the circumcircles of $\triangle AA'P$ and $\triangle BB'P$, and let their second point of intersection be Q (Q exists because of the assumptions).



Now $\angle AQA' \equiv \angle APA' \equiv BPB' \equiv BQB'$, $\angle AA'Q \equiv \angle APQ \equiv \angle BPQ \equiv \angle BB'Q$ and similarly $\angle A'AQ \equiv B'BQ$. It follows that triangles $AA'Q$ and $BB'Q$ are directly similar[4]. Now consider the spiral similarity with centre Q, angle AQA' and scale factor $\frac{A'Q}{AQ}$. It will map A to A' by construction, and from the similar triangles it will map B to B'. We can now proceed to show that S is mapped to T, as was done in the corresponding theorem for homothetisms. □

Exercise 7.6. *Squares are constructed outwards on the sides of triangle ABC. Let P, Q and R be the centres of the squares opposite A, B and C respectively. Prove that AP and QR are equal and perpendicular.*

8 Miscellaneous problems

These problems all draw on the techniques in this book, but do not fit well into any particular section. They are mostly very challenging problems designed to give you practice.

Exercise 8.1 (⋆). *ABCD is a square. P is a point inside the square with* $\angle ABP = \angle BAP = 15^\circ$*. Show that* $\triangle CDP$ *is equilateral.*

Exercise 8.2 (⋆). *A 6m tall statue stands on a pedestal, so that the foot of the statue is 2m above your head height. Determine how far from the statue you should stand so that it appears as large as possible in your vision.*[5]

[4] Two triangles are directly similar if they are similar and have the same clockwise/anti-clockwise orientation.

[5] In other words, maximise the angle formed by the foot of the statue, your head and the top of the statue.

Exercise 8.3 (⋆). *In an acute angled triangle ABC the interior bisector of $\angle A$ intersects BC at L and the circumcircle of $\triangle ABC$ again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas.*

Exercise 8.4 (⋆). *ABC is a triangle. The internal bisector of the angle A meets the circumcircle again at P. Q and R are similarly defined relative to B and C. Prove that*

$$AP+BQ+CR>AB+BC+CA.$$

Exercise 8.5 (⋆). *A circle of radius r is inscribed in a triangle ABC with area K. The points of tangency with BC, CA and AC are X, Y and Z respectively. AX intersects the circle again in X'. Prove that $BC\cdot AX\cdot XX'=4rK$.*

Exercise 8.6 (⋆). *A semicircle is drawn on one side of a straight line ℓ. C and D are points on the semicircle. The tangents at C and D meet ℓ again at B and A respectively, with the centre of the semicircle between them. Let E be the point of intersection of AC and BD, and F the point on ℓ such that EF is perpendicular to ℓ. Prove that EF bisects $\angle CFD$.*

Exercise 8.7 (⋆). *In $\triangle ABC$, let D and E be points on the side BC such that $\angle BAD=\angle CAE$. If M and N are, respectively, the points of tangency with BC of the incircles of $\triangle ABD$ and $\triangle ACE$, show that $\dfrac{1}{MB}+\dfrac{1}{MD}=\dfrac{1}{NC}+\dfrac{1}{NE}$.*

Exercise 8.8 (⋆). *Let P be a point inside $\triangle ABC$ such that*

$$\angle APB-\angle ACB=\angle APC-\angle ABC.$$

Let D, E be the incentres of $\triangle APB$, $\triangle APC$ respectively. Show that AP, BD and CE meet at a point.

9 Solutions

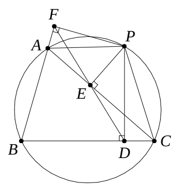
3.1 Using classical geometry to solve this problem would result in an enormous number of different cases. However, directed angles hide all of that, and the result appears with a few lines of basic calculation:

$$\begin{aligned}\angle AZC &\equiv \angle AZO+\angle OZC\\ &\equiv \angle AXO+\angle OYC && \text{(concyclic points)}\\ &\equiv \angle BXO+\angle OYB && \text{(collinear points)}\end{aligned}$$

$$\begin{aligned}&\equiv \angle BXO+\angle OXB && \text{(concyclic points)}\\ &\equiv \angle BXO-\angle BXO && \text{(directed angles)}\\ &\equiv 0^\circ,\end{aligned}$$

and hence A, Z and C are collinear.

3.2 Note that PC subtends right angles at D and E, and hence is the diameter of a circle passing through P, C, D and E. Similarly, P,A,F and E are concyclic.



$$\begin{aligned}\angle DEF &\equiv \angle DEP+\angle PEF\\ &\equiv \angle DCP+\angle PAF\\ &\equiv \angle BCP-\angle BAP.\end{aligned}$$

It follows that $\angle DEF\equiv 0^\circ \iff \angle BCP\equiv\angle BAP$. The first is a condition for D,E,F to be collinear and the second is a condition for P to lie on the circumcircle of $\triangle ABC$.

4.1 $\cot(A\pm B)=\frac{\cot A\cot B\mp 1}{\cot A\pm\cot B}$ can be shown by substituting $\tan\theta=\frac{1}{\cos\theta}$ into $\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$ and simplifying. The expressions for $\sin A\sin B$ and similar expressions can be proved simply by expanding the right hand side and cancelling terms. The final three equations are derived by making suitable substitutions into the previous three.

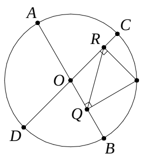
4.2 We first derive a general formula for $\tan(A+B+C)$.

$$\begin{aligned}\tan(A+B+C) &= \tan[(A+B)+C]\\ &= \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}\\ &= \frac{\frac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\frac{\tan A+\tan B}{1-\tan A\tan B}\cdot\tan C}\end{aligned}$$

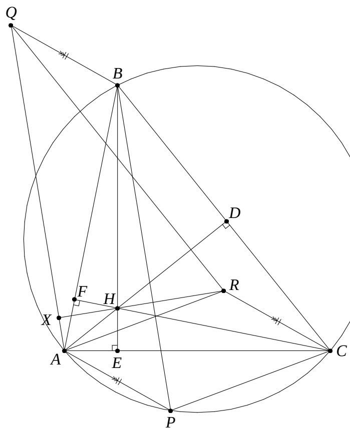
$$= \frac{\tan A + \tan B + (1 - \tan A \tan B)\tan C}{(1 - \tan A \tan B) - (\tan A + \tan B)\tan C}$$
$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

However, we know that $\tan(A+B+C) = \tan 180^\circ = 0$, so the numerator must be 0. The result follows.

4.3 Suppose that P lies on the arc BC, as in the diagram. Then $OQPR$ is cyclic with diameter OP, so applying the extended sine rule in $\triangle OQR$ gives $QR = OP \sin\angle BOC$. Now $\angle BOC$ is fixed and OP is the radius of the circle, also fixed. So QR is fixed if P lies on the arc BC. But $\sin\angle BOC = \sin\angle COA = \sin\angle DOA = \sin\angle DOB$, so QR is constant wherever on the circle P may be.



5.1 This problem is fairly straight-forward as it consists almost entirely of angle chasing. The only difficulty is that P can lie anywhere on the circumcircle, which could give rise to multiple cases. We can get around this with directed angles. This diagram is thus only for reference. D and F are the feet of the altitudes from A and C in $\triangle ABC$.



(a) Firstly notice that since $PAQB$ and $PARC$ are parallelograms, BQ and CR are parallel and equal (and in the same direction), so $BCRQ$ is also a parallelogram. It follows that $RQ \parallel CB$ and hence $AH \perp RQ$. This shows that H lies on one altitude of $\triangle AQR$. If $RX \perp AQ$ then it would lie on another altitude we would be done.

Note that B, D, H and F are concyclic. Thus

$$\begin{aligned}
\angle AHC &\equiv \angle DHF && \text{(opposite angles)}\\
&\equiv \angle DBF && (D, H, F, B \text{ concyclic})\\
&\equiv \angle CBA\\
&\equiv \angle CPA && (A, B, C, P \text{ concyclic})\\
&\equiv \angle ARC && (AP \parallel RC, AR \parallel PC)
\end{aligned}$$

and therefore A, H, R and C are concyclic. Thus

$$\begin{aligned}
\angle AXR &\equiv \angle XAR + \angle ARX\\
&\equiv \angle XAB + \angle BAC + \angle CAR + \angle ARH\\
&\equiv \angle QAB + \angle FAC + \angle ACP + \angle ACH
\end{aligned}$$

$$\begin{aligned}&\equiv \angle PBA + \angle ABP + \angle FAC + \angle ACF\\ &\equiv \angle AFC\\ &\equiv 90^\circ\end{aligned}$$

and the result follows.

(b) This is just more angle chasing, using the fact that H, X, A and E are concyclic (because of the right angles).

$$\begin{aligned}\angle AEX &\equiv \angle AHX\\ &\equiv \angle AHR\\ &\equiv \angle ACR\\ &\equiv \angle PAC\\ &\equiv \angle PAE\end{aligned}$$

from which it follows that $XE \parallel AP$.

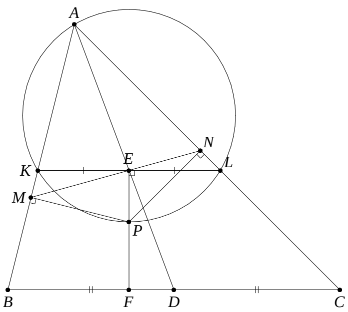
(Proposed for IMO 1996)

5.2 We use Ptolemy's Inequality:

$$\begin{aligned} & AP\cdot BC + BP\cdot CA \geq CP\cdot AB\\ \iff\quad & AP + BP \geq CP \qquad (\text{since } AP = BP = CP).\end{aligned}$$

Equality occurs if and only if $ABPC$ is a cyclic quadrilateral.

5.3 Construct KL through E parallel to BC, with K and L on AB and AC respectively.



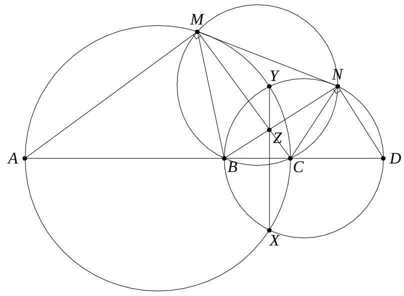
From similar triangles AKE and ABD, we have $KE = BD\cdot\frac{AE}{AD}$. Similarly, $EL = DC\cdot\frac{AE}{AD}$. But $BD = DC$, so $KE = EL$ and hence AE is a median of $\triangle AKL$. Also, $PE \perp KL$ (since $KL \parallel BC$), so M, E and N are the pedal points of P in triangle AKL. The Simpson Line theorem states that M, E and N are collinear if and only if P lies on the circumcircle of $\triangle AKL$. But the perpendicular bisector of KL and the angle bisector of $\angle A$ both meet the circumcircle at the middle of the arc KL, so P lies on the angle bisector of $\angle A$.

(*Crux Mathematicorum*, 1990, 293)

5.4 If P is one of the midpoints, then the lengths of the tangents from P to the two circles are equal. Since these lengths are the square roots of the power of P with respect to these two circles, P must lie on the radical axis. Since this is true for four midpoints, they are collinear because the radical axis is a straight line.

5.5 Call the given circles Γ_1 and Γ_2, and construct a third circle Γ_3 which intersects both Γ_1 and Γ_2. The position of Γ_3 is arbitrary, provided that the centres of the three circles are not collinear. The radical axes of (Γ_1, Γ_2) and (Γ_1, Γ_3) can be found by drawing lines through the intersection points. The intersection of these two lines is the radical centre of the three circles. The desired radical axis now passes through the radical centre and is perpendicular to the line of centres of Γ_1 and Γ_2, which can easily be constructed.

5.6 We use directed angles and line segments, since P may lie either inside or outside of the segment XY. It is also possible (but more tedious) to do the proof with two cases. The diagram below shows the one case.



Label the circle with diameter AC as Γ_1, and the circle with diameter BD as Γ_2. The point Z lies on the radical axis of the two circles, so it has equal power with respect to both. In particular, $ZM \cdot ZC = ZN \cdot ZB$, which prove that M, N, B and C are concyclic. Call this circle Γ_3. Now

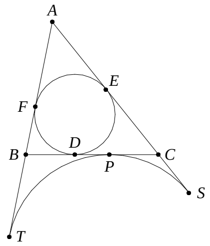
$$\begin{aligned}\angle MND &\equiv \angle MNB + \angle BND \\ &\equiv \angle MCB + 90^\circ \\ &\equiv \angle MCA + \angle AMC \\ &\equiv -\angle CAM \\ &\equiv \angle MAD.\end{aligned}$$

This proves that M, N, A and D are also concyclic; call this circle Γ_4. Finally, we note that AM, DN and XY are the three radical axes formed between the circles Γ_1, Γ_2 and Γ_4. These lines are not all parallel ($AM \parallel XY$ would require that $P = Z$), so they must coincide at the radical centre of the circles.

(IMO 1995, problem 1)

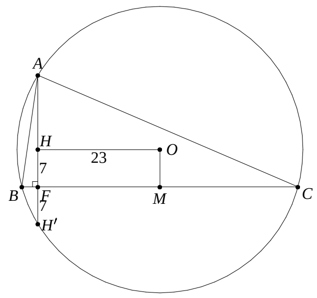
6.1 Let D, E and F be the points of tangency of the incircle with BC, CA, AB and let the excircle be tangent to the same sides at P, S and T respectively. Then from common tangents,

$$\begin{aligned}2ES = 2FT &= ES + FT \\ &= EC + CS + FB + BT \\ &= DC + CP + DB + BP \\ &= 2BC.\end{aligned}$$



Hence $ES = FT = BC = y + z$. Now $BP = BT = FT - BF = (y + z) - y = z$. Similarly, $CP = y$.

6.2 Since the altitude AF passes through H and $BC \perp AF$, it follows that BC and FM coincide. Let H' be the reflection of H in BC. H' is known to lie on the circumcircle of $\triangle ABC$, so $R = H'O = \sqrt{23^2 + 14^2}$. Hence $BM = \sqrt{H'O^2 - 7^2} = 26$ and $BC = 2BM = 52$.



6.3 Clearly, A_0, B_0 and C_0 are in fact I_A, I_B and I_C, and we will refer to them as such.

(i) We will show that $|\triangle II_AC| = 2|\triangle IA_1C|$ (refer to the diagram on page 15, where D is A_1). Results for five other pairs of triangles follow similarly, and adding them all up gives the desired result. Triangles II_AC and IA_1C have a common height, and bases II_A and IA_1. But these bases are the radius and diameter of the circle with diameter II_A, so the result follows.

(ii) It suffices to show that $|AC_1BA_1CB_1|$ is at least twice $|ABC|$, which is equivalent to showing that $|\triangle BCA_1| + |\triangle CAB_1| + |\triangle ABC_1| \geq |\triangle ABC|$. Let A_2, B_2 and C_2 be the reflections of H in BC, CA and AB. These points are known to lie on the circumcircle. When comparing the areas of triangles BCA_1 and BCA_2, we note that they share a common base but the height of $\triangle BCA_1$ is greater than or equal to that of $\triangle BCA_2$. Hence

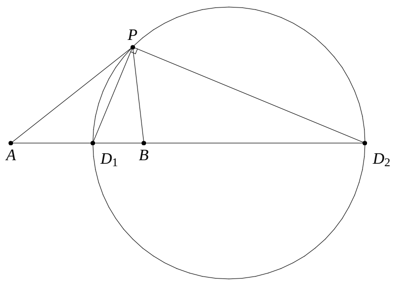
$$\begin{aligned}|\triangle BCA_1| + |\triangle CAB_1| + |\triangle ABC_1| &\geq |\triangle BCA_2| + |\triangle CAB_2| + |\triangle ABC_2| \\ &= |\triangle BCH| + |\triangle CAH| + |\triangle ABH| \\ &= |\triangle ABC|.\end{aligned}$$

(IMO 1989 Question 2)

6.4 Let $BD = m$ and $DC = n$. Then $m+n = a$ and $\frac{n}{m} = \frac{a-m}{m} = \frac{b}{c}$. Hence

$$BD = m = \frac{a}{1+\frac{b}{c}} = \frac{ac}{b+c}.$$

6.5 If $r = 1$, then $AP = BP$ and so the locus is simply the perpendicular bisector. Otherwise suppose $r > 1$ (the situation is symmetric if $r < 1$). Pick an arbitrary P not on AB which satisfies the condition. Let the internal and external angle bisectors of $\angle APB$ meet AB at D_1 and D_2 respectively. Then by the angle bisector theorem, $\frac{AD_1}{BD_1} = \frac{AD_2}{BD_2} = r$. D_1 and D_2 are the only two points on AB that satisfy this, so they are fixed independent of P. Also, $\angle D_1PD_2 = 90°$, so P must lie on the circle with diameter D_1D_2.

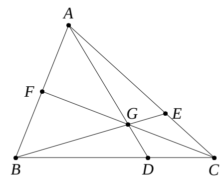


Conversely, suppose P lies on this circle. If P also lies on AB then $P = D_1$ or $P = D_2$, both of which satisfy the conditions. Otherwise let the internal and external bisectors of $\angle PAB$ meet AB at E_1 and E_2 respectively. If $\frac{AP}{BP} = \frac{AE_1}{BE_1} = \frac{AE_2}{BE_2} < r$ then E_1 lies closer to A than D_1 and E_2 lies further from A than D_2. But this means that $\angle E_1PE_2 > 90°$, which is a contradiction. Similarly, if $\frac{AP}{BP} > r$ then $\angle E_1PE_2 < 90°$, again a contradiction. Thus $\frac{AP}{BP} = r$, and this circle is precisely the locus of P.

This circle is known as an *Apollonius* circle.

6.6 Apply Menelaus to $\triangle ACD$ cut by line BGE:

$$\frac{AG}{GD} \cdot \frac{DB}{BC} \cdot \frac{CE}{EA} = -1. \quad (9.1)$$

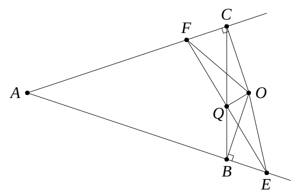


Similarly, one can apply it to $\triangle ABD$ cut by BGE:

$$\frac{AG}{GD} \cdot \frac{DC}{CB} \cdot \frac{BF}{FA} = -1. \quad (9.2)$$

Finally, dividing (9.1) by (9.2) and doing some re-arranging (while being careful with the sign conventions) gives Ceva's Theorem.

6.7 Without loss of generality, let $BQ \leq CQ$, giving the diagram below:



Suppose $OQ \perp EF$. Then $EBQO$ and $FCOQ$ are cyclic quadrilaterals, so $\angle BEO = 180° - \angle BQO = \angle CQO = \angle CFO$. But $BO = CO$, so $\triangle BEO \equiv \triangle CFO$. This gives $EO = FO$, making $\triangle EOF$ isosceles. But $OQ \perp EF$, so $EQ = QF$.

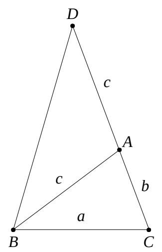
Now suppose that $QE = QF$. Apply Menelaus to triangle AEF, cut by line BQC:

$$1 = \frac{EQ}{QF} \cdot \frac{FC}{CA} \cdot \frac{AB}{BE} = \frac{FC}{BE}.$$

Hence $CF = BE$. Also, $BO = CO$, so $\triangle BEO \equiv \triangle CFO$ and hence $EO = FO$. Then $\triangle EOF$ is isosceles with $EQ = QF$, so $OQ \perp EF$.

(IMO 1994 question 2)

6.8 Construct D on the extension of AC such that $\angle ABD = \angle ABC$. Note that AB is then an angle bisector of $\triangle BDC$. Also, $\angle BDA = 2\angle ABC - \angle ABD = \angle ABD$, so triangle ABD is isosceles and $AD = c$. From the angle bisector theorem (or from $\triangle ABC \,|||\, \triangle BDC$), we find that $AD = \frac{ac}{b}$.



From Stewart's Theorem, we get

$$\begin{aligned} & (b+c)(c^2+bc) = \left(\frac{ac}{b}\right)^2 \cdot b + a^2c \\ \implies \quad & (b+c)^2bc = a^2c^2 + a^2bc \\ \implies \quad & b(b+c) = a^2, \end{aligned}$$

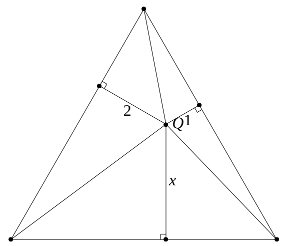
as required.

6.9 Let the orthic triangle be $A'B'C'$. We use Euler's Inequality twice, once on $\triangle ABC$ and once on $\triangle A'B'C'$. The vertices of the orthic triangle lie on the nine-point circle, so the circumradius of $\triangle A'B'C'$ is $R/2$. Thus

$$\begin{aligned} \frac{p}{R} &= \frac{1}{2} \cdot \frac{p}{R/2} \\ &\leq \frac{1}{4} \\ &= 1 - \frac{1}{3} \cdot \frac{3^2}{2} \\ &\leq 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2. \end{aligned}$$

(Proposed at IMO 1993)

6.10 The height of the triangle is 6, so the area is $12\sqrt{3}$. Let the required length be x, and consider the area as the sum of the areas of the triangles formed by Q and the vertices.



The total area is thus $2\sqrt{3}(1+2+x)$. Solving the equation $12\sqrt{3} = 2\sqrt{3}(1+2+x)$ gives $x = 3$.

6.11 We know that $s = x+y+z$. Divide through by K, recalling that $K = rs = r_ax = r_by = r_cz$.

6.12 We first check that sin is concave on $[0°, 180°]$:

$$\frac{\sin x + \sin y}{2} = \sin\left(\frac{x+y}{2}\right) \cdot \cos\left(\frac{x-y}{2}\right) \leq \sin\left(\frac{x+y}{2}\right).$$

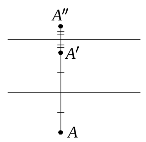
Thus

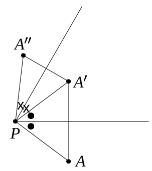
$$\sin\alpha + \sin\beta + \sin\gamma \leq 3\sin\left(\frac{\alpha+\beta+\gamma}{3}\right) = 3\sin 60° = \frac{3\sqrt{3}}{2}.$$

6.13 Suppose for a contradiction that these angles are all strictly greater than 30°. Drop perpendiculars from P onto BC, CA, AB to meet at D, E, F respectively. Then $2PF > PA$, $2PD > PB$ and $2PE > PC$. But then $PA+PB+PC < 2(PD+PE+PF)$, which contradicts the Erdős-Mordell Theorem.

(IMO 1991, question 5)

7.1 (a) When combining two reflections, there are two cases.





In the diagrams above, the first reflection maps A to A', and the second maps A' to A''.

(i) The lines of reflection are parallel, separated by a distance d. As can be seen from the diagram, the combination of the reflections is a translation by $2d$, perpendicular to the lines of reflection (the direction depends on the order in which the reflections are performed. Conversely, any translation can be expressed as the combination of two parallel reflections, suitably oriented, and with separation equal to half the distance of the translation.

(ii) The lines of reflection are not parallel, and intersect at some point P with an angle of θ. From the diagram, it is now clear that any other point is rotated by an angle of 2θ around P, with the direction depending on the order of the rotations. Conversely, any rotation can be expressed as the combination of two reflections which pass through the centre of the rotation, and with an angle between them of half the rotation angle.

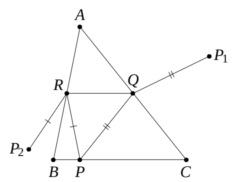
(b) Two translations trivially produce another translation, whose displacement is the vector sum of the original displacements. When one or both of the transformations is a rotation, express the transformations as pairs of reflections. We showed in part (a) that there is some freedom in the choice of reflections. We will have four reflections which are applied in order, say $b_2b_1a_2a_1$.[6] We can always choose the reflections such that a_2 and b_1 are the same. Identical reflections cancel out, so we are left with a_1b_2 which from (a) is equivalent to a rotation or translation.

(c) We can transform all the rotations and translations into pairs of reflections, using part (a). We can then pair off these reflections and convert them back into translations and rotations, possibly leaving one reflection at the end. Now part (b) shows that we can reduce the sequence of translations and

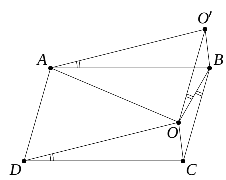
[6] We write sequence of transformations from right to left. This is because they are functions, so applying ab to a point P actually means $a(b(P))$, with b being applied first.

rotations to just one, which may be followed by a reflection. It remains to show that a rotation followed by a reflection is equivalent to a translation followed by a reflection. We do this by appending two identical (and hence cancelling) reflections to the sequence, at an angle we will choose in a moment. The sequence will now appear as $ccb(r_2r_1)$ where r_2r_1 is the rotation, and c is the newly added reflection. We choose c so that cb forms a rotation with angle exactly opposite to the angle of r_2r_1. Now $(cb)(r_2r_1)$ is the combination of two rotations that forms some translation, say T (it is a translation, not a rotation, because of the choice of angle). Thus the entire sequence is equivalent to cT i.e. a translation followed by a reflection.

7.2 Reflect P in CA to obtain P_1 and reflect P in AB to obtain P_2. Now $PQ+QR+RP = P_1Q+QR+RP_2$. This sum will clearly be smallest when P_1, Q, R and P_2 lie in a straight line. So choose Q and R to be the intersections of P_1P_2 with CA and AB.



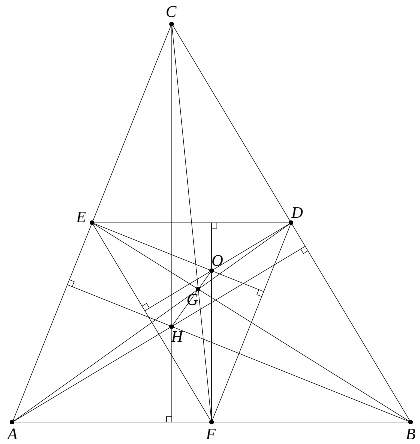
7.3 Having two supplementary angles vertically opposite each other is not very helpful. It would be more useful if we could get the angles to be either adjacent (to create a straight line) or opposite angles of a quadrilateral (to make it cyclic). One way to do this is to "pick up" triangle DOC and place DC on top of AB.



More formally, construct O' outside $ABCD$ such that $\triangle AO'B \equiv \triangle DOC$. Then $\angle AO'B + \angle AOB = 180^\circ$, so $AO'BO$ is cyclic. Also, $OO'BC$ is a parallelogram because $O'B$ and OC are equal and parallel. Thus $\angle OBC = \angle BOO' = \angle BAO' = \angle ODC$.

(Canadian Mathematical Olympiad 1997)

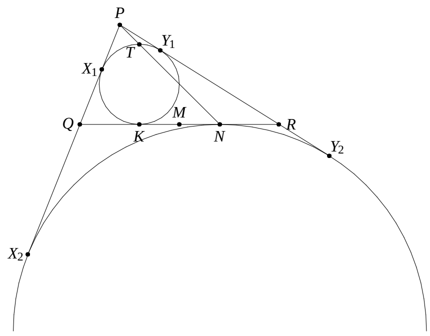
7.4 (a) Let D, E and F be the midpoints of BC, CA and AB respectively. From the Midpoint Theorem, $\triangle DEF \parallel\!| \triangle ABC$ and is half the size. It is also oriented 180° relative to $\triangle ABC$. Thus there is a homothetism that maps $\triangle ABC$ to $\triangle DEF$, with scale factor $-\frac{1}{2}$. The centre of similitude must lie on AD, BE and CF, and hence these lines are concurrent.



(b) The homothetism maps AG to DG with scale factor $-\frac{1}{2}$, so $AG : GD = 2 : 1$. The result follows similarly for the other two medians.

(c) The line DO is perpendicular to BC, and hence also to EF. Similarly $EO \perp FD$ and $FO \perp DE$, so O is the orthocentre of $\triangle DEF$. Since the homothetism maps $\triangle ABC$ to $\triangle DEF$, it will also map H to O. This proves the collinearity, and the scale follows as in the previous section.

7.5 Start with an arbitrary pair (Q, R) for which P exists, and construct the excircle C_2 of $\triangle PQR$ opposite P (see diagram).



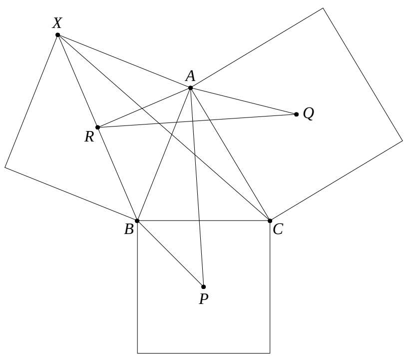
The incircle and excircle of $\triangle PQR$ must be homothetic, and P is the centre of the homothetism. Now let K be the point of tangency of C with L, and let T be the point diametrically opposite K. The corresponding point to T on C_2 must also be vertically above the centre in the diagram, i.e. it is N. But the line through corresponding points must pass through the centre of the homothetism, so P lies on NT.

From the solution to problem 6.1 (page 37), we have $QK = RN$, from which it follows that N and K are symmetrically placed about M. But K and M are fixed, so N must be fixed too.

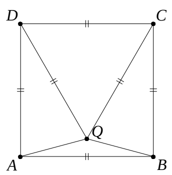
We have now established that any solution P must lie on NT. It is also clear that P must lie strictly beyond T. Conversely, suppose P' is some point on NT beyond T. Let L' be a line through P' and parallel to L, and consider moving a point P along L', finding Q and R on L such that C is the incircle of $\triangle PQR$. When P moves far to the left, the midpoint of QR will be far to the right, and vice versa. Since the midpoint shifts continuously, there is at least one point where it is M. We have shown above that this P must be the intersection of NT with L', namely P', and hence P' satisfies the desired properties. Therefore the locus is the portion of NT that lies strictly beyond T.

(IMO 1992, question 4)

7.6 Consider the spiral similarity with centre A, rotating clockwise (in the diagram) by $45°$ and scaling by $\sqrt{2}$. It will map Q to C and R to X. Now consider the spiral similarity with centre B that rotates anti-clockwise by $45°$ and scales by $\sqrt{2}$. It will map A to X and P to C. These two similarities thus map AP and QR to the same line. They both scale by the same amount ($\sqrt{2}$) and the difference of their angles is $90°$, so AP and QR must be equal and perpendicular.



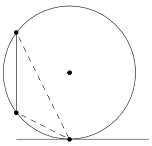
8.1 Construct Q inside the square with $\triangle CDQ$ equilateral. We aim to show that $P = Q$.



Now $\angle QDC = 60°$, so $\angle QDA = 30°$. But $QD = AD$, so $\triangle AQD$ is isosceles and thus $\angle DAQ = 75°$. This makes $\angle BAQ = 15°$, and similarly $\angle ABQ = 15°$. But then triangles ABP and ABQ have two common angles and a common side, so they are congruent. Both P and Q lie on the same side of AB (the inside

of the square), so P and Q must be the same. Triangle CDQ is equilateral by construction, so $\triangle CDP$ is equilateral.

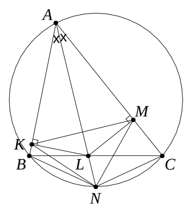
8.2 Construct a circle of radius 5m, with centre 5m above your head height and 4m from the statue. This circle will pass through the head and foot of the statue.



If your head lies on the circle you will have some constant viewing angle θ; with your head inside the circle the angle is larger, and with your head outside the circle it is smaller. But the circle is tangent to the line representing head-height, so the best angle is when your head is at this point of tangency. So you should stand 4m from the statue.

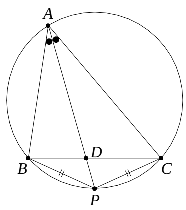
8.3 Firstly note that $\triangle ALK \equiv \triangle ALM$. Hence $AKLM$ is a kite and so $KM \perp AL$; thus $|AKNM| = \frac{1}{2}KM \cdot AN$. Since $ABNC$ is cyclic, $\triangle ABL \,|||\, \triangle ANC$ and hence $AN \cdot AL = AB \cdot AC$. Also, AL is the diameter of the circumcircle of $\triangle AKM$, so $\frac{KM}{AL} = \sin\alpha$. Substituting these into the above gives

$$\begin{aligned} |AKNM| &= \frac{1}{2} \cdot \frac{KM \cdot AB \cdot AC}{AL} \\ &= \frac{1}{2} \cdot AB \cdot AC \cdot \sin\alpha \\ &= |\triangle ABC| \end{aligned}$$



(IMO 1987 Question 2)

8.4 Let D be the point where the angle bisector from A cuts BC.

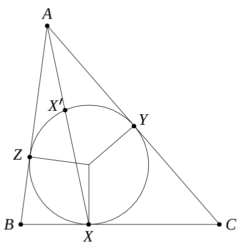


Since $\angle BAD = \angle PAC$ and $\angle DBA = \angle CPA$ we have $\triangle BAD ||| \triangle PAC$. Thus $\frac{c}{BD} = \frac{AP}{PC}$. From exercise 6.4 we have $BD = \frac{ac}{b+c}$. It follows that $AP = \frac{b+c}{a} \cdot PC$. But $PB = PC$ and so from the triangle inequality, $2PC > BC \iff PC > \frac{a}{2}$. Therefore $AP > \frac{b+c}{2}$.

Similarly $BQ > \frac{c+a}{2}$ and $CR > \frac{a+b}{2}$. Adding these inequalities gives the desired result.

(Australian Mathematics Olympiad 1985)

8.5 Firstly note that $AX \cdot AX'$ is the power of A with respect to the incircle, so it is equal to $AZ^2 = x^2$. Thus $a \cdot AX \cdot XX' = a \cdot AX^2 - ax^2$.



We can calculate $a \cdot AX^2$ using Stewart's Theorem:

$$BC(AX^2 + BX \cdot XC) = AC^2 \cdot BX + AB^2 \cdot CX$$

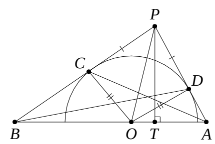
$$\begin{aligned} a(AX^2 + yz) &= b^2 y + c^2 z \\ a \cdot AX^2 &= (x+z)^2 y + (x+y)^2 z - (y+z)yz \\ &= x^2 y + 2xyz + z^2 y + x^2 z + 2xyz + y^2 z - y^2 z - z^2 y \\ &= x^2(y+z) + 4xyz \\ &= ax^2 + 4xyz. \end{aligned}$$

Now we can calculate $a \cdot AX^2 - ax^2$

$$\begin{aligned} a \cdot AX^2 - ax^2 &= 4xyz \\ &= \frac{4}{s} \cdot sxyz \\ &= \frac{4}{s} \cdot K^2 \\ &= \frac{4}{s} \cdot rsK \\ &= 4rK \text{ as desired.} \end{aligned}$$

(Arbelos May 1987)

8.6 This is a good example of a problem that becomes much easier with a good diagram (the diagram below is intentionally skewed). If AD and BC are extended to meet at P, then it appears that P, E and F are collinear. This would be a useful thing to know, so we attempt to prove it.



Let T be the foot of the perpendicular from P to AB and let O be the centre of the semicircle. $\triangle OCB ||| \triangle PTB$, so $\frac{CB}{TB} = \frac{BO}{BP}$. Similarly $\frac{DA}{TA} = \frac{AO}{AP}$. We want to prove that PT, AC and BD are concurrent, which by the converse of Ceva's Theorem would be true if

$$\frac{PC}{CB} \cdot \frac{BT}{TA} \cdot \frac{AD}{DP} = 1$$

Firstly, $PC = PD$ (equal tangents to the semicircle), and we can substitute the ratios found above to change this to $\frac{BP}{BO} \cdot \frac{AQ}{AP} = 1$. However, this is true by the angle bisector theorem (PD is an angle bisector because $\triangle PCO \equiv \triangle PDO$). It follows that E lies on the altitude from A, and $F = T$.

Now notice that PO subtends right angles at C, D and F, so $PCFD$ is a cyclic quad. Thus $\angle DFP = \angle DCP$ and $\angle CFP = \angle CDP$, and since $PC = PD$ it follows that $\angle DFP = \angle CFP$. Therefore EF bisects $\angle CFD$.

(Proposed at IMO 1994)

8.7 The key to this problem is noticing that you can treat triangles ABD and ACE as completely separate, and ignore $\triangle ABC$. The only things these two triangles have in common is the angle at A and the height from A. Let these quantities be θ and h respectively. If we can express $\frac{1}{MB} + \frac{1}{MD}$ in terms of θ and h then we are done.

Let us rename D to C so that we are working with $\triangle ABC$ and can use the usual notation.

$$\begin{aligned}
\frac{1}{MB} + \frac{1}{MC} &= \frac{1}{y} + \frac{1}{z} \\
&= \frac{y+z}{yz} \\
&= \frac{a}{yz} \\
&= \frac{ahrsx}{hrsxyz} \\
&= \frac{ahrsx}{hrK^2} \quad \text{(Heron's Formula)} \\
&= \frac{2K^2x}{hrK^2} \\
&= \frac{x}{r} \cdot \frac{2}{h} \\
&= \frac{2}{h} \cot \frac{\theta}{2}.
\end{aligned}$$

(Proposed at IMO 1993)

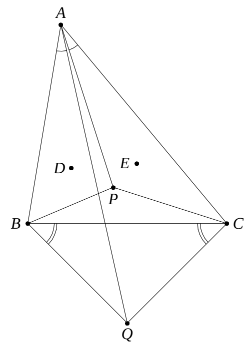
8.8 Construct Q so that $\angle BAQ = \angle PAC$ and $\angle ABQ = \angle APC$. Then by construction, $\triangle ABQ \,|||\, \triangle APC$. Now in $\triangle APB$ and $\triangle ACQ$:

- $\angle BAP = \angle BAC - \angle PAC = \angle QAC$

- $\frac{AC}{AQ} = \frac{AC}{AC \cdot AB/AP} = \frac{AP}{AB}$.

Hence $\triangle APB \,|||\, \triangle ACQ$. Now $\angle CBQ = \angle APC - \angle ABC = \angle APB - \angle ACB = \angle BCQ$, so $\triangle BCQ$ is isosceles. It follows that

$$\frac{AC}{PC} = \frac{AQ}{BQ} = \frac{AQ}{CQ} = \frac{AB}{BP}.$$



Now from the angle bisector theorem, BD will cut AP in the ratio $AB : BP$, and CE will cut AP in the ratio $AC : CP$. Since these ratios are the same, the three lines will be concurrent.

(IMO 1996 Question 2)

10 Recommended further reading

Geometric inequalities often require techniques from the world of standard inequalities. *Inequalities for the Olympiad Enthusiast*, by Graeme West (part of the same series as this booklet) provides some good material in this field.

This booklet is well under 100 pages, and as such cannot do proper justice to the rich field of classical geometry. A highly regarded and very readable reference is *Geometry Revisited*, by Coxeter and Greitzer.

A good source of problems are the yearbooks of the South African training program for the IMO (*South Africa and the* n*th IMO*, for $n \geq 35$). These contain problems

Geometry : metric properties

1. General trigonometry

   - $\sin(X \pm Y) = \sin X \cos Y \pm \sin Y \cos X$
   - $\cos(X \pm Y) = \cos X \cos Y \mp \sin X \sin Y$
   - $\tan(X \pm Y) = \frac{\tan X \pm \tan Y}{1 \mp \tan X \tan Y}$
   - $\sin 2X = 2 \sin X \cos X$
   - $\cos 2X = \cos^2 X - \sin^2 X = 2\cos^2 X - 1 = 1 - 2\sin^2 X$
   - $\sin X = \frac{2t}{1+t^2}, \ \cos X = \frac{1-t^2}{1+t^2}, \ \tan X = \frac{2t}{1-t^2} \qquad (t = \tan \frac{X}{2})$
   - $\sin P \pm \sin Q = 2 \sin \frac{P \pm Q}{2} \cos \frac{P \mp Q}{2}$
   - $\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}, \ \cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}$

2. If $A + B + C = 180°$, then

   - $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
   - $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
   - $\tan(kA) + \tan(kB) + \tan(kC) = \tan(kA)\tan(kB)\tan(kC) \qquad (k \in \mathbb{Z})$
   - $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
   - $\cos 2A + \cos 2B + \cos 2C = -(1 + 4 \cos A \cos B \cos C)$
   - $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$
   - $\sin^2 A + \sin^2 B + \sin^2 C = 2.(1 + \cos A \cos B \cos C)$
   - $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$

3. Area of a triangle

$$
\begin{aligned}
S &= \frac{a.h_A}{2} = \frac{b.h_B}{2} = \frac{c.h_C}{2} \\
&= \frac{ab}{2} \sin C = \frac{bc}{2} \sin A = \frac{ca}{2} \sin B \\
&= pr \\
&= (p-a)r_A = (p-b)r_B = (p-c)r_C \\
&= \sqrt{p(p-a)(p-b)(p-c)} \\
&= \sqrt{r.r_A.r_B.r_C} \\
&= \frac{abc}{4R} \\
&= 2R^2 \sin A \sin B \sin C \\
&= 4rR \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\
&= p^2 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}
\end{aligned}
$$

4. Other relations

- $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$
- $a^2 = b^2 + c^2 - 2bc\cos A,\ b^2 = c^2 + a^2 - 2ca\cos B,\ c^2 = a^2 + b^2 - 2ab\cos C$
- $\frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \cos A + \cos B + \cos C - 1$
- $\frac{r_A}{R} = 4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},\ \frac{r_B}{R} = 4\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2},\ \frac{r_C}{R} = 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
- $\frac{r}{p-a} = \frac{r_A}{p} = \tan\frac{A}{2},\ \frac{r}{p-b} = \frac{r_B}{p} = \tan\frac{B}{2},\ \frac{r}{p-c} = \frac{r_C}{p} = \tan\frac{C}{2}$
- $\frac{r}{p} = \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$
- $r = a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\frac{\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}} = c\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$.
- $r_A + r_B + r_C - r = 4R$
- $\frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C} - \frac{1}{r} = 0$

5. Distances between vertices and tangency points

	Incircle		
	T_A	T_B	T_C
A	•	$p-a$	$p-a$
B	$p-b$	•	$p-b$
C	$p-c$	$p-c$	•

	Excircles								
	T_{AA}	T_{AB}	T_{AC}	T_{BA}	T_{BB}	T_{BC}	T_{CA}	T_{CB}	T_{CC}
A	•	p	p	•	$p-c$	$p-c$	•	$p-b$	$p-b$
B	$p-c$	•	$p-c$	p	•	p	$p-a$	•	$p-a$
C	$p-b$	$p-b$	•	$p-a$	$p-a$	•	p	p	•

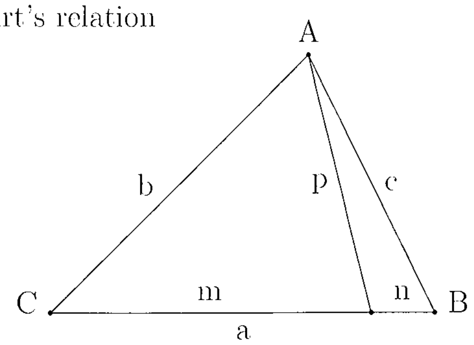
6. Other distances and powers with respect to the circumcircle

X	$\lvert OX\rvert^2$	$\pi_\Gamma(X) = \lvert OX\rvert^2 - R^2$
H	$9R^2 - (a^2+b^2+c^2)$	$-8R^2\cos A\cos B\cos C$
G	$R^2 - \frac{1}{9}(a^2+b^2+c^2)$	$-\frac{1}{9}(a^2+b^2+c^2)$
E	$\frac{9}{4}R^2 - \frac{1}{4}(a^2+b^2+c^2)$	$\frac{5}{4}R^2 - \frac{1}{4}(a^2+b^2+c^2)$
I	$R^2 - 2Rr$	$-2Rr$
I_A	$R^2 + 2Rr_A$	$2Rr_A$
I_B	$R^2 + 2Rr_B$	$2Rr_B$
I_C	$R^2 + 2Rr_C$	$2Rr_C$

7. Feuerbach's Theorem : Γ_E is internally tangent to γ and externally tangent to γ_A, γ_B and γ_C. Therefore, we have

$$|EI| = \frac{R}{2} - r,\ |EI_A| = \frac{R}{2} + r_A,\ |EI_B| = \frac{R}{2} + r_B,\ |EI_C| = \frac{R}{2} + r_C.$$

8. Stewart's relation



$$mc^2 + nb^2 = ap^2 + mn^2 + nm^2$$

9. Menelaus' and Ceva's Theorems : if $D \in BC$, $E \in CA$ and $F \in AB$, then
   - AD, BE and CF are concurrent iff $\frac{\overline{DB}}{\overline{DC}} \cdot \frac{\overline{EC}}{\overline{EA}} \cdot \frac{\overline{FA}}{\overline{FB}} = -1$ (Ceva)
   - AD, BE and CF are concurrent iff $\frac{\sin \widehat{DAB}}{\sin \widehat{DAC}} \cdot \frac{\sin \widehat{EBC}}{\sin \widehat{EBA}} \cdot \frac{\sin \widehat{FCA}}{\sin \widehat{FCB}} = -1$ (trig Ceva)
   - D, E and F are collinear iff $\frac{\overline{DB}}{\overline{DC}} \cdot \frac{\overline{EC}}{\overline{EA}} \cdot \frac{\overline{FA}}{\overline{FB}} = 1$ (Menelaus)

   (Note: "concurrent" means "concurrent or parallel")

10. Miquel point and pedal triangle : if $D \in BC$, $E \in CA$ and $F \in AB$, then
    - $AEMF$, $BFMD$ and $CDME$ are cyclic quadrilaterals iff

    $$\widehat{AMB} = \widehat{ACB} + \widehat{EFD}, \ \widehat{BMC} = \widehat{BAC} + \widehat{FDE}, \ \widehat{CMA} = \widehat{CBA} + \widehat{DEF}$$

    In this case (with $\delta = \widehat{AEF} = \widehat{BFD} = \widehat{CDE}$), we have

    $$|DE| = \frac{|MC| \cdot \sin C}{\sin \delta}, \quad |EF| = \frac{|MA| \cdot \sin A}{\sin \delta} \quad \text{and} \quad |FD| = \frac{|MB| \cdot \sin B}{\sin \delta}$$

    - DEF is the pedal triangle of M (i.e. $\delta = 90°$) iff

    $$|DB|^2 + |EC|^2 + |FA|^2 = |DC|^2 + |EA|^2 + |FB|^2$$

vervolg hiervan

Dit laatste komt door het volgende:

Stelling 1.4. *(stellingen van Carnot)*

- *Zij O het omcentrum van $\triangle ABC$ dan geldt dat de som van de afstanden van O tot de middens van de zijden van de driehoek gelijk is aan $R+r$ waarbij een afstand afgetrokken werd als ze buiten de driehoek ligt.*
- *X, Y, Z zijn punten op de rechten AB, AC, BC van driehoek $\triangle ABC$.*

  De loodlijnen uit deze 3 punten zijn concurrent a.e.s.a.

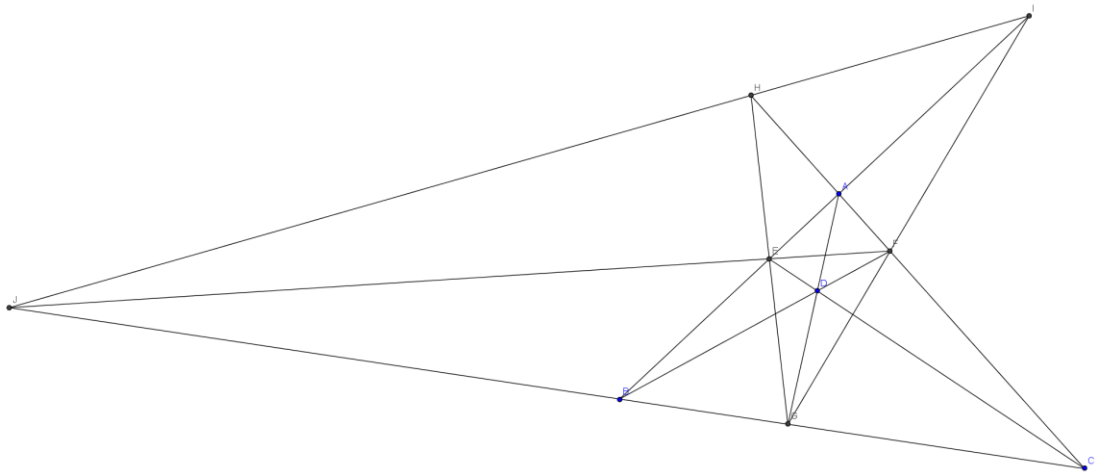
  $|BX|^2 + |AY|^2 + |ZC|^2 = |BZ|^2 + |CY|^2 + |AX|^2$
- *Een triviale stelling van Carnot: A, B, X, Y zijn 4 punten, dan geldt dat $AB \perp XY$ $\Leftrightarrow AX^2 - BX^2 = AY^2 - BY^2$.*

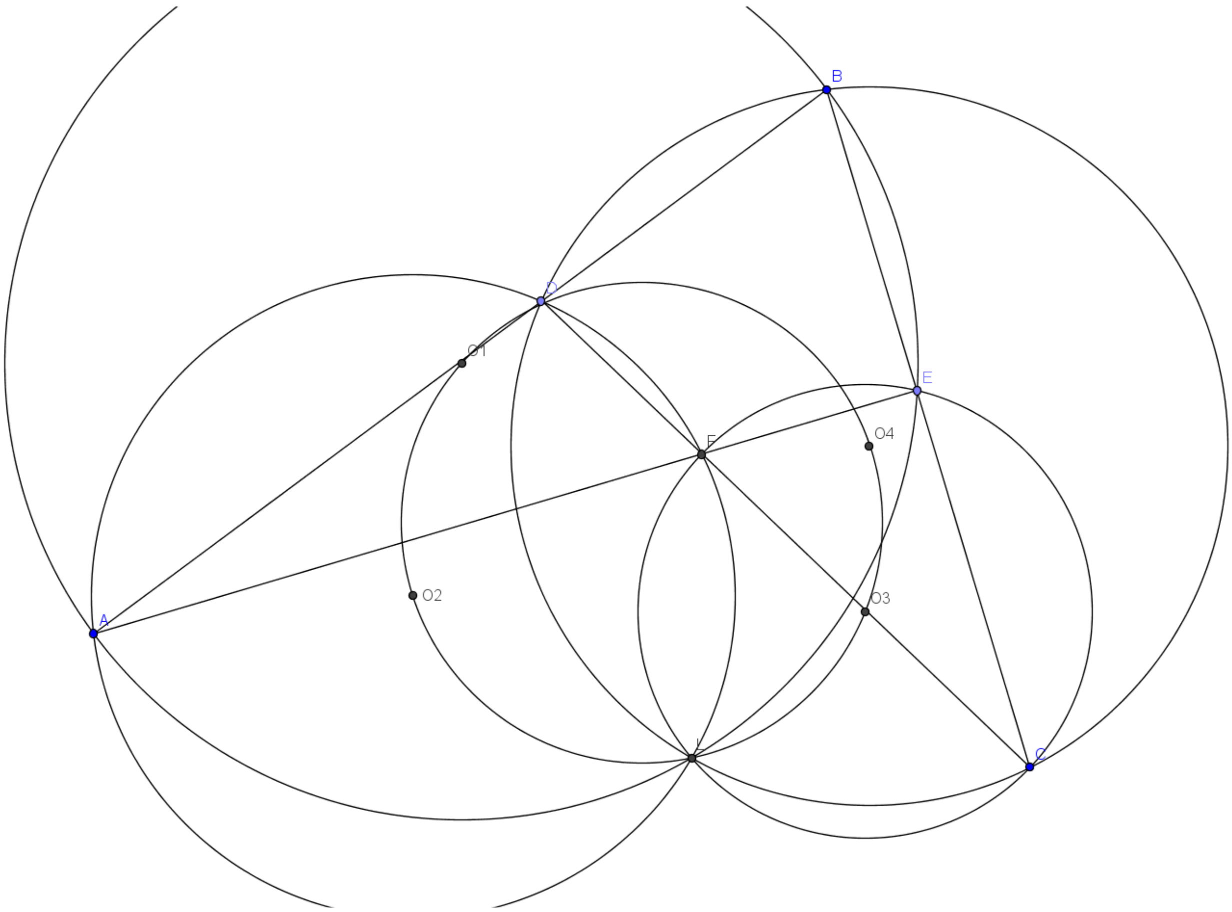
extra eigenschap *In ΔABC tekent men de voetpuntsdriehoek DEF van P. Er geldt dat*

- $\text{Opp(DEF)} = \frac{1}{4}\left|1 - \frac{|OP|^2}{R^2}\right| \cdot \text{Opp(ABC)}.$

Stelling 1.5. *(stellingen van Miquel bij vierhoeken)*

M is het Miquel punt tov AB, CD, AC, BD wanneer geldt dat X het snijpunt is van de omgeschreven cirkels , van de snijpunten van ieder triplet lijnen. Stel $P = AC \cap BD, R = AB \cap CD$. Er geldt met de stelling dat M op de omgeschreven cirkels van ABP, CPD, RBD en RAC ligt. De hoogtelijnen van de 4 driehoeken liggen op een rechte (gevolg van de steinerlijn). De 4 omcentra zijn cyclisch met het Miquelpunt




B
D
O1
E
F
O4
O2
A
O3
C

1.1 vervollediging naamgeving

- de trilineaire pool en poollijn van een driehoek: zie bovenste tekening op vorige bladzijde. In een driehoek ABC is er een punt P (de trilineaire pool), AP, BP, CP snijden hun overstaande zijden in $A", B", C"$ dan snijden $A'B', AB$ en analoog op een rechte, deze noemen we de trilineaire poollijn (bestaat door de stelling van Desargues)

- ceviaandriehoek:

  $\triangle A"B"C"$ is de ceviaandriehoek van $\triangle ABC$ tov het punt P als

  $A", B", C"$ gelijk zijn aan $AP \cup BC, BP \cup AC$ en $CP \cup AB$ resp.

- anticeviaandriehoek, als $\triangle A'B'C'$ is de anticeviaandriehoek van $\triangle ABC$ tov het punt P als

  A op de rechte $B'C'$ ligt en zo ook $C', A,' B$ collineair zijn en $C \in A'B'$

  $P = AA' \cup BB' \cup CC'$

  $\triangle ABC$ de ceviaandriehoek is van $A'B'C'$ tov P

  Er geldt dat $(AA"PA') = -1$ en de cyclische viertallen zijn harmonisch (zie sectie deelverhoudingen).

  Enkele anticeviaandriehoeken van belangrijke punten:

  I : excentral triangle

  Z: anticomplementaire driehoek

  symmediaanpunt: raaklijnendriehoek (driehoek gevormd door de raaklijnen)

- circumceviaandriehoek / circumceviantriangle

  In $\triangle ABC$ is P een inwendig punt (geen hoekpunt), dan is $A'B'C'$ de circumceviaandriehoek als $A' = PA \cup \tau$ en analoog met τ de omgescreven cirkel. Als $O = P$ is A' de antipode / diametraal overgesteld punt van A.

- contactdriehoek (intouchtriangle) voetpuntsdriehoek van I

- aanraakdriehoek: driehoek gevormd door de raakpunten van $\triangle ABC$ met zijn aangeschreven cirkels

- isogonaal verwant: 2 punten zijn isogonaal verwant als de rechten door deze punten en een hoekpunt van de driehoek, symmetrisch zijn tov de bissectrice uit dat hoekpunt en dit voor ieder hoekpunt (hun overeenkomstige cevianen/ hoektransversalen hebben een omgekeerde verhouding tot de zijden)

  De voetpuntsdriehoeken van 2 isogonaal verwante punten liggen op 1 cirkel, waarvan het middelpunt het midden is van die 2 punten.

- isotomisch verwantschap: 2 punten zijn isotomischl verwant als de loodlijnen op een zijde van de driehoek, symmetrisch zijn tov de middelloodlijnen uit dat hoekpunt en dit voor iedere zijde (vb. het punt van Gergonne en het punt van Nagel zijn isotomisch verwant)

- symmedianen: De symmedianen zijn de spiegelbeelden van de zwaartelijnen in de bissectrices =de rechten isogonaal verwant met de zwaartelijnen
- Het punt van Lemoine in een driehoek is het snijpunt van de symmedianen. Het punt van Lemoine is het punt dat de som van de kwadraten van de afstanden tot de zijden van de driehoek minimaliseert en is isogonaal verwant met het zwaartepunt.
- het punt van Nagel: X, Y, Z zijn de raakpunten van I_a, I_b, I_c met de driehoek $\triangle ABC$, dan is het N_a het punt van concurrenctie van AX, BY en CZ.
- rechte van Nagel: N_a, I, S (punt van Spieker dat het incentrum is van de middendriehoek) en Z liggen op deze rechte in de verhouding $|NS| : |SZ| : |ZI| = 3 : 1 : 2$.
- rechte van Euler: H, N, Z, O op 1 rechte liggen in die volgorde en de verhoudingen zijn: $HN : NZ : ZO = 3 : 1 : 2$
- het inwendig punt van Gergonne: het snijpunt van de lijnen door de hoekpunten en de raakpunten van de ingeschreven cirkel aan de overstaande zijden. uitwendig punt van Gergonne: snijpunten van cevianen door de raakpunten van 1 aangeschreven cirkel
- de Longchapspunt L: het hoogtepunt v.d. anticomplementaire driehoek en is het punt zodat O het midden is van HL.
- Bevanpunt: omcentrum V van de aangeschreven driehoek/excentral triangle $I_aI_bI_c$ ($\triangle$ gevormd door aancentra)

  voetpuntsdriehoek van V is de aanraakdriehoek

  V = het midden van $[NaL]$ en ligt er samen met V' (isogonaal verwante punt van V')

  O is het midden van $[IV]$

  het spiekerpunt is het midden van $[HV]$
- Het punt van Fermat:

  In het geval dat de grootste hoek van de driehoek kleiner is dan 120, is de totale afstand van het punt naar de drie hoekpunten minimaal.

  De binnenste hoeken, gevormd door dit punt: $\angle AFB, \angle BFC, \angle CFA$ zijn alle gelijk aan 120°

  De omschreven cirkels van de drie gelijkzijdige driehoeken van de constructie snijden in dit punt

  De driehoek, gevormd door de centra van de drie gelijkzijdige driehoeken in de constructie is ook een gelijkzijdige driehoek (Stelling van Napoleon) en het centrum van de omgeschreven cirkel van deze driehoek is het punt van Fermat van de originele driehoek

  Als $\triangle ABZ, \triangle ACY, \triangle BCX$ de uitwendige gelijkzijdige driehoeken zijn, geldt dat $F = CZ \cup BY \cup AX$. F is het punt waarvoor $AP+BP+CP$ minimaal is, bewijs door Z' te nemen door P 60 te draaien rond A richting Z, waarna $PZ' = AP$ en $|ZZ'| = |PB|$.

- Brocardpunten: zij $O1$ het eerste punt van Brocard van driehoek $\triangle ABC$, dan geldt dat $\angle O1AB = \angle O1BC = \angle O1CA = \gamma = \angle ABO_2 = \angle BCO_2 = \angle CAO_2$ met O_2 het tweede Brocardpunt, dat het isogonaal geconjugeerd punt van O_1 is.
- de Steinerlijn is de lijn l gevormd door een punt P op de omgeschreven cirkel te spiegelen over AB, BC, AC en gaat door H. Het is de homothetie met center in P van de Simsonlijn met een factor 2. Het punt P wordt het antiSteinerpunt van l tov $\triangle ABC$ genoemd

gevolgen van stelling van Menelaos

Stelling 1.6. *(stelling van Monge)*

De stelling van Monge-d'Alembert zegt dat als we 3 cirkels hebben, de gemeenschappelijke uitwendige raaklijnen snijden in 3 punten die collineair zijn.

Hierbij bedoelen we dat de uitwendige raaklijnen van Γ_i en γ_{i+1} snijden in P_i en dat P_1, P_2, P_3 op 1 rechte liggen.

alternatieve vorm:

Zij O_1, O_2, O_3 de 3 centra van resp. $\Gamma_1, \Gamma_2, \Gamma_3$, zij P_1 het snijpunt van de uitwendige raaklijnen van Γ_1, Γ_2 en P_2, P_3 de snijpunten van de inwendige raaklijnen van resp. Γ_3, Γ_2 en Γ_1, Γ_3. Dan geldt dat P_1, P_2, P_3 opnieuw collineair zijn.

Stelling 1.7. *(transversaalstelling)*

A, B, C zijn 3 punten op een lijn en P is een punt die niet op die lijn ligt: A', B', C' zijn 3 punten die liggen op AP, BP, CP resp., dan geldt er dat de punten A', B', C' collineair zijn als en slechts als $\frac{\vec{BC}\vec{AP}}{\vec{A'P}} + \frac{\vec{CA}\vec{BP}}{\vec{B'P}} \cdot \frac{\vec{AB} \cdot \vec{CP}}{\vec{C'P}} = 0$

Voorbeeld 1.8. *(Darij Grinberg, uitdaging) $\triangle ABC$ is een willekeurige driehoek met P, P' 2 punten in het vlak. $A' = AP \cup BC, B' = BP \cup CA$ en $A" = AP' \cup BC, B" = BP' \cup CA$. Q, Q' zijn de isogonaal geconjugeerde punten van P, P' resp. tov $\triangle ABC$. TB: $Q \in A"B" \Leftrightarrow Q' \in A'B'$.*

Stelling 1.9. *(Menelaos voor vierhoeken)*

Als X, Y, Z, W punten zijn op AB, BC, CD, AD van een vierhoek $ABCD$ en deze 4 punten liggen op een rechte, dan geldt dat

$\frac{\vec{AX}}{\vec{XB}} \cdot \frac{\vec{BY}}{\vec{YC}} \cdot \frac{\vec{CZ}}{\vec{ZD}} \cdot \frac{\vec{DW}}{\vec{WA}} = 1$

(deze stelling geldt niet in beide richtingen)

Stelling 1.10. *(Cevian Nests theorem)*

$\triangle ABC$ is een willekeurige driehoek met A', B', C' 3 punten op BC, AC, AB resp. en $A", B", C"$ 3 punten op de rechten $B'C', A'C', A'B'$ resp., dan gelden de volgende 3 uitspraken als er 2 waar zijn:

- AA', BB', CC' *zijn concurrent*
- $AA", BB", CC"$ *zijn concurrent*
- $A'A", B'B", C'C"$ *zijn concurrent*

gevolgen van stelling van Ceva

Stelling 1.11. *(de goniometrische vorm van Ceva)*

Zij D, E, F punten die resp. op AB, AC, BC liggen, dan geldt dat AD, BE, CF concurrent zijn a.e.s.a.

$\frac{sin(BAF)\cdot sin(ACD)\cdot sin(CBE)}{sin(DCB)\cdot sin(FAC)\cdot sin(EBA)} = 1$

Stelling 1.12. *(stelling van Jacobi)*

Gegegen een driehoek $\triangle ABC$ en 3 punten X, Y, Z die alle 3 inwendig zijn of alle 3 uitwendig liggen). Als $\angle ZAB = \angle YAC, \angle ZBA = \angle XBC$ en $\angle XCB = \angle YCA$ dan zijn de lijnen AX, BY, CZ concurrent.

opmerking: ook te bewijzen met inversie en machtlijnen

2 de lemma's

Lemmas in Euclidean Geometry[1]

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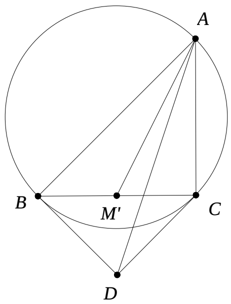
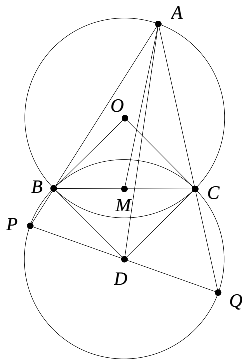
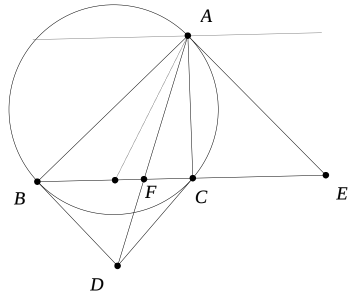
1. **Construction of the symmedian.**

Let ABC be a triangle and Γ its circumcircle. Let the tangent to Γ at B and C meet at D. Then AD coincides with a symmedian of $\triangle ABC$. (The *symmedian* is the reflection of the median across the angle bisector, all through the same vertex.)







We give three proofs. The first proof is a straightforward computation using Sine Law. The second proof uses similar triangles. The third proof uses projective geometry.

First proof. Let the reflection of AD across the angle bisector of $\angle BAC$ meet BC at M'. Then

$$\frac{BM'}{M'C} = \frac{AM'\frac{\sin\angle BAM'}{\sin\angle ABC}}{AM'\frac{\sin\angle CAM'}{\sin\angle ACB}} = \frac{\sin\angle BAM'}{\sin\angle ACD}\frac{\sin\angle ABD}{\sin\angle CAM'} = \frac{\sin\angle CAD}{\sin\angle ACD}\frac{\sin\angle ABD}{\sin\angle BAD} = \frac{CD}{AD}\frac{AD}{BD} = 1$$

Therefore, AM' is the median, and thus AD is the symmedian. □

Second proof. Let O be the circumcenter of ABC and let ω be the circle centered at D with radius DB. Let lines AB and AC meet ω at P and Q, respectively. Since $\angle PBQ = \angle DQC + \angle BAC = \frac{1}{2}(\angle BDC + \angle DOC) = 90^\circ$, we see that PQ is a diameter of ω and hence passes through D. Since $\angle ABC = \angle AQP$ and $\angle ACB = \angle APQ$, we see that triangles ABC and AQP are similar. If M is the midpoint of BC, noting that D is the midpoint of QP, the similarity implies that $\angle BAM = \angle QAD$, from which the result follows. □

Third proof. Let the tangent of Γ at A meet line BC at E. Then E is the pole of AD (since the polar of A is AE and the pole of D is BC). Let BC meet AD at F. Then point B, C, E, F are harmonic. This means that line AB, AC, AE, AF are harmonic. Consider the reflections of the four line across the angle bisector of $\angle BAC$. Their images must be harmonic too. It's easy to check that AE maps onto a line parallel to BC. Since BC must meet these four lines at harmonic points, it follows that the reflection of AF must pass through the midpoint of BC. Therefore, AF is a symmedian. □

[1]Updated July 26, 2008

Related problems:

(i) (Poland 2000) Let ABC be a triangle with $AC = BC$, and P a point inside the triangle such that $\angle PAB = \angle PBC$. If M is the midpoint of AB, then show that $\angle APM + \angle BPC = 180°$.

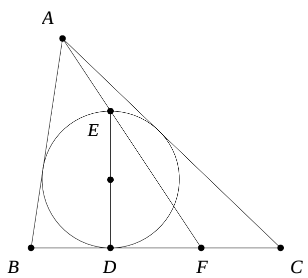
(ii) (IMO Shortlist 2003) Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose center does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

(iii) (Vietnam TST 2001) In the plane, two circles intersect at A and B, and a common tangent intersects the circles at P and Q. Let the tangents at P and Q to the circumcircle of triangle APQ intersect at S, and let H be the reflection of B across the line PQ. Prove that the points A, S, and H are collinear.

(iv) (USA TST 2007) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lies on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

(v) (USA 2008) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

2. **Diameter of the incircle.**



Let the incircle of triangle ABC touch side BC at D, and let DE be a diameter of the circle. If line AE meets BC at F, then $BD = CF$.

Proof. Consider the dilation with center A that carries the incircle to an excircle. The diameter DE of the incircle must be mapped to the diameter of the excircle that is perpendicular to BC. It follows that E must get mapped to the point of tangency between the excircle and BC. Since the image of E must lie on the line AE, it must be F. That is, the excircle is tangent to BC at F. Then, it follows easily that $BD = CF$. □

Related problems:

(i) (IMO Shortlist 2005) In a triangle ABC satisfying $AB + BC = 3AC$ the incircle has centre I and touches the sides AB and BC at D and E, respectively. Let K and L be the symmetric points of D and E with respect to I. Prove that the quadrilateral $ACKL$ is cyclic.

(ii) (IMO 1992) In the plane let $\mathcal{C}$ be a circle, ℓ a line tangent to the circle $\mathcal{C}$, and M a point on ℓ. Find the locus of all points P with the following property: there exists two points Q, R on ℓ such that M is the midpoint of QR and $\mathcal{C}$ is the inscribed circle of triangle PQR.

(iii) (USAMO 1999) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

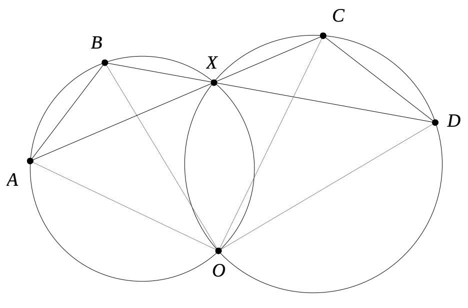
(iv) (USAMO 2001) Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

(v) (Tournament of Towns 2003 Fall) Triangle ABC has orthocenter H, incenter I and circumcenter O. Let K be the point where the incircle touches BC. If IO is parallel to BC, then prove that AO is parallel to HK.

(vi) (IMO 2008) Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

3. **Dude, where's my spiral center?**

Let AB and CD be two segments, and let lines AC and BD meet at X. Let the circumcircles of ABX and CDX meet again at O. Then O is the center of the spiral similarity that carries AB to CD.



Proof. Since $ABOX$ and $CDXO$ are cyclic, we have $\angle OBD = \angle OAC$ and $\angle OCA = \angle ODB$. It follows that triangles AOC and BOD are similar. The result is immediate. □

Remember that spiral similarities always come in pairs: if there is a spiral similarity that carries AB to CD, then there is one that carries AC to BD.

Related problems:

(i) (IMO Shortlist 2006) Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle CBA = \angle DCA = \angle EDA.$$

Diagonals BD and CE meet at P. Prove that line AP bisects side CD.

(ii) (China 1992) Convex quadrilateral $ABCD$ is inscribed in circle ω with center O. Diagonals AC and BD meet at P. The circumcircles of triangles ABP and CDP meet at P and Q. Assume that points O, P, and Q are distinct. Prove that $\angle OQP = 90°$.

(iii) Let $ABCD$ be a quadrilateral. Let diagonals AC and BD meet at P. Let O_1 and O_2 be the circumcenters of APD and BPC. Let M, N and O be the midpoints of AC, BD and O_1O_2. Show that O is the circumcenter of MPN.

(iv) (USAMO 2006) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

(v) (IMO 2005) Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

(vi) (IMO Shortlist 2002) Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

(vii) (USA TST 2006) In acute triangle ABC, segments AD, BE, and CF are its altitudes, and H is its orthocenter. Circle ω, centered at O, passes through A and H and intersects sides AB and AC again at Q and P (other than A), respectively. The circumcircle of triangle OPQ is tangent to segment BC at R. Prove that $CR/BR = ED/FD$.

(viii) (IMO Shortlist 2006) Points A_1, B_1 and C_1 are chosen on sides BC, CA, and AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, and CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, and C_2, respectively ($A_2 \neq A, B_2 \neq B$, and $C_2 \neq C$). Points A_3, B_3, and C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of sides BC, CA, and AB, respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

4. **Arc midpoints are equidistant to vertices and in/excenters**

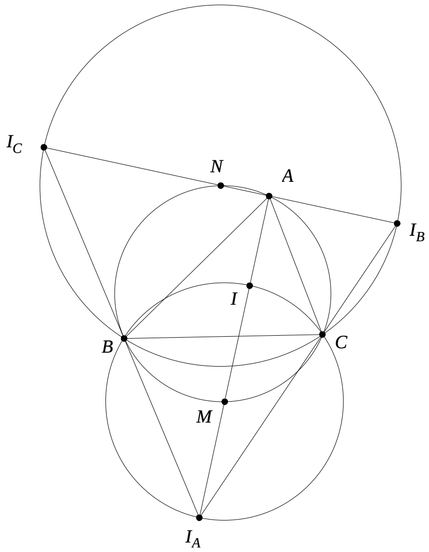
Let ABC be a triangle, I its incenter, and I_A, I_B, I_C its excenters. On the circumcircle of ABC, let M be the midpoint of the arc BC not containing A and let N be the midpoint of the arc BC containing A. Then $MB = MC = MI = MI_A$ and $NB = NC = NI_B = NI_C$.

Proof. Straightforward angle-chasing (do it yourself!). Another perspective is to consider the circumcircle of ABC as the nine-point-circle of $I_AI_BI_C$. □

Related problems:

(i) (APMO 2007) Let ABC be an acute angled triangle with $\angle BAC = 60°$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

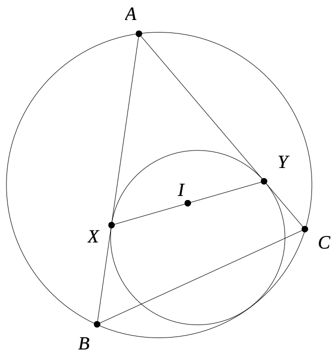
(ii) (IMO 2006) Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

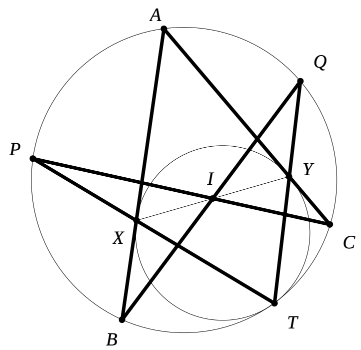


(iii) (Romanian TST 1996) Let $ABCD$ be a cyclic quadrilateral and let $\mathcal{M}$ be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (16 points in total). Prove that there are two sets $\mathcal{K}$ and $\mathcal{L}$ of four parallel lines each, such that every line in $\mathcal{K} \cup \mathcal{L}$ contains exactly four points of $\mathcal{M}$.

5. **I is the midpoint of the touch-chord of the mixtilinear incircles**

Let ABC be a triangle and I its incenter. Let Γ be the circle tangent to sides AB, AC, as well as the circumcircle of ABC. Let Γ touch AB and AC at X and Y, respectively. Then I is the midpoint of XY.





Proof. Let the point of tangency between the two circles be T. Extend TX and TY to meet the circumcircle of ABC again at P and Q respectively. Note that P and Q are the midpoint of the arcs AB and AC. Apply Pascal's theorem to $BACPTQ$ and we see that X, I, Y are collinear. Since I lies on the angle bisector of $\angle XAY$ and $AX = AY$, I must be the midpoint of XY. ☐

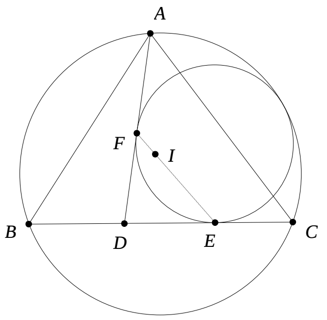
Related problems:

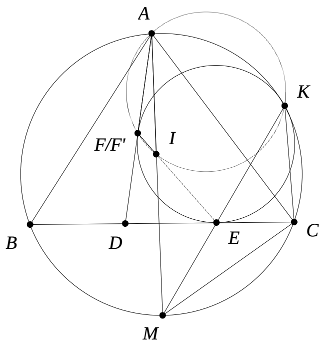
(i) (IMO 1978) In triangle ABC, $AB = AC$. A circle is tangent internally to the circumcircle of triangle ABC and also to sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC.

(ii) Let ABC be a triangle. Circle ω is tangent to AB and AC, and internally tangent to the circumcircle of triangle ABC. The circumcircle and ω are tangent at P. Let I be the incircle of triangle ABC. Line PI meets the circumcircle of ABC at P and Q. Prove that $BQ = CQ$.

6. **More curvilinear incircles.**

(A generalization of the previous lemma) Let ABC be a triangle, I its incenter and D a point on BC. Consider the circle that is tangent to the circumcircle of ABC but is also tangent to DC, DA at E, F respectively. Then E, F and I are collinear.





Proof. There is a "computational" proof using Casey's theorem[2] and transversal theorem[3]. You can try to work that out yourself. Here, we show a clever but difficult synthetic proof (communicated to me via Oleg Golberg).

Denote Ω the circumcircle of ABC and Γ the circle tangent tangent to the circumcircle of ABC and lines DC, DA. Let Ω and Γ touch at K. Let M be the midpoint of arc $\widehat{BC}$ on Ω not containing K. Then K, E, M are collinear (think: dilation with center K carrying Γ to Ω). Also, A, I, M are collinear, and $MI = MC$.

Let line EI meet Γ again at F'. It suffices to show that AF' is tangent to Γ.

Note that $\angle KF'E$ is subtended by $\widehat{KE}$ in Γ and $\angle KAM$ is subtended by $\widehat{KM}$ in Ω. Since $\widehat{KE}$ and $\widehat{KM}$ are homothetic with center K, we have $\angle KF'E = \angle KAM$, implying that A, K, I', F' are concyclic.

We have $\angle BCM = \angle CBM = \angle CKM$. So $\triangle MCE \sim \triangle MKC$. Hence $MC^2 = ME \cdot MK$. Since $MC = MI$, we have $MI^2 = ME \cdot MK$, implying that $\triangle MIE \sim \triangle MKI$. Therefore,

[2]**Casey's theorem**, also known as Generalized Ptolemy Theorem, states that if there are four circles $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ (could be degenerated into a point) all touching a circle Γ such that their tangency points follow that order around the circle, then

$$t_{12}t_{34} + t_{23}t_{14} = t_{13}t_{24},$$

where t_{12} is the length of the common tangent between Γ_i and Γ_j (if Γ_i and Γ_j on the same side of Γ, then take their common external tangent, else take their common internal tangent.) I think the converse is also true—if both equations hold, then there is some circle tangent to all four circles.

[3]The **transversal theorem** is a criterion for collinearity. It states that if A, B, C are three collinear points, and P is a point not on the line ABC, and A', B', C' are arbitrary points on lines PA, PB, PC respectively, then A', B', C' are collinear if and only if

$$BC \cdot \frac{AP}{A'P} + CA \cdot \frac{BP}{B'P} + AB \cdot \frac{CP}{C'P} = 0,$$

where the lengths are directed. In my opinion, it's much easier to remember the proof than to memorize this huge formula. The simplest derivation is based on relationships between the areas of $[PAB]$, $[PA'B']$, etc.

$\angle KEI = \angle AIK = \angle AF'K$ (since A, K, I, F' are concyclic). Therefore, AF' is tangent to Ω and the proof is complete. □

Related problems:

(i) (Bulgaria 2005) Consider two circles k_1, k_2 touching externally at point T. A line touches k_2 at point X and intersects k_1 at points A and B. Let S be the second intersection point of k_1 with the line XT . On the arc $\widehat{TS}$ not containing A and B is chosen a point C . Let CY be the tangent line to k_2 with $Y \in k_2$, such that the segment CY does not intersect the segment ST . If $I = XY \cap SC$. Prove that:

(a) the points C, T, Y, I are concyclic.

(b) I is the excenter of triangle ABC with respect to the side BC.

(ii) (Sawayama-Thébault[4]) Let ABC be a triangle with incenter I. Let D a point on side BC. Let P be the center of the circle that touches segments AD, DC, and the circumcircle of ABC, and let Q be the center of the circle that touches segments AD, BD, and the circumcircle of ABC. Show that P, Q, I are collinear.

(iii) Let P be a quadrilateral inscribed in a circle Ω, and let $\mathcal{Q}$ be the quadrilateral formed by the centers of the fourcircles internally touching O and each of the two diagonals of P. Show that the incenters of the four triangles having for sides the sides and diagonals of P form a rectangle R inscribed in Q.

(iv) (Romania 1997) Let ABC be a triangle with circumcircle Ω, and D a point on the side BC. Show that the circle tangent to Ω, AD and BD, and the circle tangent to Ω, AD and DC, are tangent to each other if and only if $\angle BAD = \angle CAD$.

(v) (Romania TST 2006) Let ABC be an acute triangle with $AB \neq AC$. Let D be the foot of the altitude from A and ω the circumcircle of the triangle. Let ω_1 be the circle tangent to AD, BD and ω. Let ω_2 be the circle tangent to AD, CD and ω. Let ℓ be the interior common tangent to both ω_1 and ω_2, different from CD. Prove that ℓ passes through the midpoint of BC if and only if $2BC = AB + AC$.

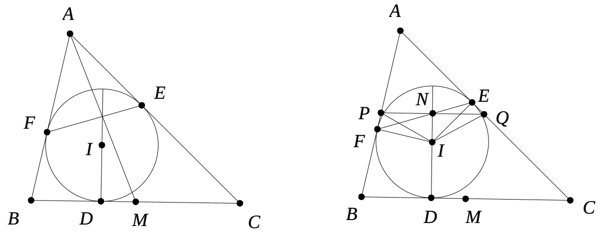
(vi) (AMM 10368) For each point O on diameter AB of a circle, perform the following construction. Let the perpendicular to AB at O meet the circle at point P. Inscribe circles in the figures bounded by the circle and the lines AB and OP. Let R and S be the points at which the two incircles to the curvilinear triangles AOP and BOP are tangent to the diameter AB. Show that $\angle RPS$ is independent of the position of O.

7. **Concurrent lines from the incircle.**

Let the incircle of ABC touch sides BC, CA, AB at D, E, F respectively. Let I be the incenter of ABC and M be the midpoint of BC. Then the lines EF, DI and AM are concurrent.

Proof. Let lines DI and EF meet at N. Construct a line through N parallel to BC, and let it meet sides AB and AC at P and Q, respectively. We need to show that A, N, M are collinear, so it suffices to show that N is the midpoint of PQ. We present two ways to finish this off, one using Simson's line, and the other using spiral similarities.

[4]A bit of history: this problem was posed by French geometer Victor Thébault (1882–1960) in the *American Mathematical Monthly* in 1938 (Problem 2887, 45 (1938) 482–483) and it remained unsolved until 1973. However, in 2003, Jean-Louis Ayme discovered that this problem was independently proposed and solved by instructor Y. Sawayama of the Central Military School of Tokyo in 1905! For more discussion, see Ayme's paper at `http://forumgeom.fau.edu/FG2003volume3/FG200325.pdf`



Simson line method: Consider the triangle APQ. The projections of the point I onto the three sides of APQ are D, N, F, which are collinear, I must lie on the circumcircle of APQ by Simson's theorem. But since AI is an angle bisector, $PI = QI$, thus $PN = QN$.

Spiral similarity method: Note that P, N, I, F are concyclic, so $\angle EFI = \angle QPI$. Similarly, $\angle PQI = \angle FEI$. So triangles PIQ and FIE are similar. Since $FI = EI$, we have $PI = QI$, and thus $PN = QN$. (c.f. Lemma 3) □

Related problems:

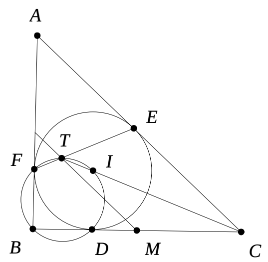
(i) (China 1999) In triangle ABC, $AB \neq AC$. Let D be the midpoint of side BC, and let E be a point on median AD. Let F be the foot of perpendicular from E to side BC, and let P be a point on segment EF. Let M and N be the feet of perpendiculars from P to sides AB and AC, respectively. Prove that M, E, and N are collinear if and only if $\angle BAP = \angle PAC$.

(ii) (IMO Shortlist 2005) The median AM of a triangle ABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y. The lines AX and AY intersect BC at P and Q. Prove that $BP = CQ$.

8. **More circles around the incircle.**

Let I be the incenter of triangle ABC, and let its incircle touch sides BC, AC, AB at D, E and F, respectively. Let line CI meet EF at T. Then T, I, D, B, F are concyclic. Consequent results include: $\angle BTC = 90°$, and T lies on the line connecting the midpoints of AB and BC.

An easier way to remember the third part of the lemma is: for a triangle ABC, draw a midline, an angle bisector, and a touch-chord, each generated from different vertex, then the three lines are concurrent.



Proof. Showing that I, T, E, B are concyclic is simply angle chasing (e.g. show that $\angle BIC = \angle BFE$). The second part follows from $\angle BTC = \angle BTI = \angle BFI = 90°$. For the third part, note that if M is the midpoint of BC, then M is the midpoint of an hypotenuse of the right triangle BTC. So $MT = MC$. Then $\angle MTC = \angle MCT = \angle ACT$, so MT is parallel to AC, and so MT is a midline of the triangle. □

Related problems:

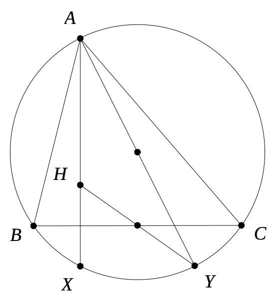
(i) Let ABC be an acute triangle whose incircle touches sides AC and AB at E and F, respectively. Let the angle bisectors of $\angle ABC$ and $\angle ACB$ meet EF at X and Y, respectively, and let the midpoint of BC be Z. Show that XYZ is equilateral if and only if $\angle A = 60°$.

(ii) (IMO Shortlist 2004) For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

(iii) Let points A and B lie on the circle Γ, and let C be a point inside the circle. Suppose that ω is a circle tangent to segments AC, BC and Γ. Let ω touch AC and Γ at P and Q. Show that the circumcircle of APQ passes through the incenter of ABC.

9. **Reflections of the orthocenter lie on the circumcircle.**

Let H be the orthocenter of triangle ABC. Let the reflection of H across the BC be X and the reflection of H across the midpoint of BC be Y. Then X and Y both lie on the circumcircle of ABC. Moreover, AY is a diameter of the circumcircle.



Proof. Trivial. Angle chasing. □

Related problems:

(i) Prove the existence of the nine-point circle. (Given a triangle, the nine-point circle is the circle that passes through the three midpoints of sides, the three feet of altitudes, and the three midpoints between the orthocenter and the vertices).

(ii) Let ABC be a triangle, and P a point on its circumcircle. Show that the reflections of P across the three sides of ABC lie on a lie that passes through the orthocenter of ABC.

(iii) (IMO Shortlist 2005) Let ABC be an acute-angled triangle with $AB \neq AC$, let H be its orthocentre and M the midpoint of BC. Points D on AB and E on AC are such that $AE = AD$ and D, H, E are collinear. Prove that HM is orthogonal to the common chord of the circumcircles of triangles ABC and ADE.

(iv) (USA TST 2005) Let $A_1A_2A_3$ be an acute triangle, and let O and H be its circumcenter and orthocenter, respectively. For $1 \leq i \leq 3$, points P_i and Q_i lie on lines OA_i and $A_{i+1}A_{i+2}$ (where $A_{i+3} = A_i$), respectively, such that OP_iHQ_i is a parallelogram. Prove that

$$\frac{OQ_1}{OP_1} + \frac{OQ_2}{OP_2} + \frac{OQ_3}{OP_3} \geq 3.$$

(v) (China TST quizzes 2006) Let ω be the circumcircle of triangle ABC, and let P be a point inside the triangle. Rays AP, BP, CP meet ω at A_1, B_1, C_1, respectively. Let A_2, B_2, C_2 be the images of A_1, B_1, C_1 under reflection about the midpoints of BC, CA, AB, respectively. Show that the orthocenter of ABC lies on the circumcircle of $A_2B_2C_2$.

10. **O and H are isogonal conjugates.**

Let ABC be a triangle, with circumcenter O, orthocenter H, and incenter I. Then AI is the angle bisector of $\angle HAO$.

Proof. Trivial. □

Related problems:

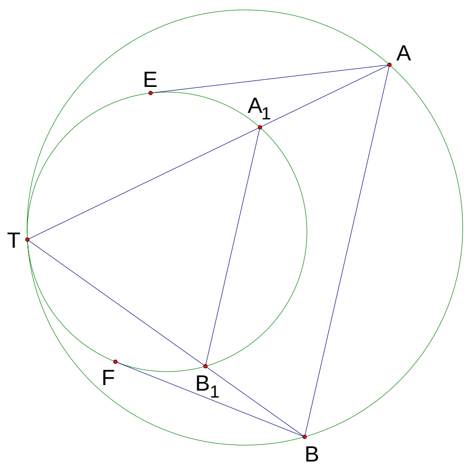
(i) (Crux) Points O and H are the circumcenter and orthocenter of acute triangle ABC, respectively. The perpendicular bisector of segment AH meets sides AB and AC at D and E, respectively. Prove that $\angle DOA = \angle EOA$.

(ii) Show that $IH = IO$ if and only if one of $\angle A$, $\angle B$, $\angle C$ is $60°$.

A Metric Relation and its Applications

Son Hong Ta

Lemma. Let γ be a circle and let A and B be two arbitrary points on it. A circle ρ touches γ internally at T. Denote by AE and BF the tangent lines to ρ at E and F, respectively. Then $\frac{TA}{TB} = \frac{AE}{BF}$.



Proof. Denote by A_1 and B_1 the second intersections of TA and TB with ρ, respectively. We know that A_1B_1 is parallel to AB. Therefore,

$$\left(\frac{AE}{TA_1}\right)^2 = \frac{AA_1 \cdot AT}{A_1T \cdot A_1T} = \frac{BB_1}{B_1T} \cdot \frac{BT}{B_1T} = \left(\frac{BF}{TB_1}\right)^2.$$

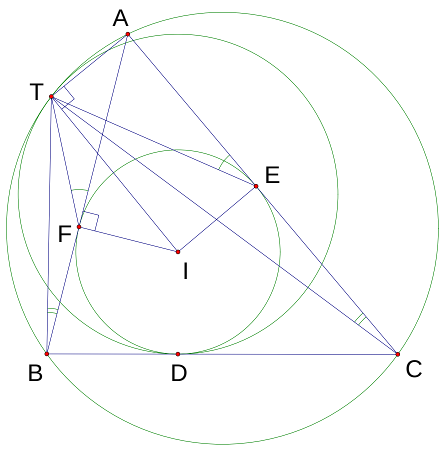
Hence,

$$\frac{AE}{TA_1} = \frac{BF}{TB_1} \implies \frac{AE}{BF} = \frac{TA_1}{TB_1} = \frac{TA}{TB},$$

which completes the proof. □

To illustrate how this lemma works, let us consider some examples. The following problem was proposed by Nguyen Minh Ha, in the Vietnamese Mathematics Magazine, in 2007.

Problem 1. Let Ω be the circumcircle of the triangle ABC and let D be the tangency point of its incircle $\rho(I)$ with the side BC. Let ω be the circle internally tangent to Ω at T, and to BC at D. Prove that $\angle ATI = 90°$.



Solution. Let E and F be the tangency points of $\rho(I)$ with sides CA and AB, respectively. According to the lemma,

$$\frac{TB}{TC} = \frac{BD}{CD} = \frac{BF}{CE}.$$

Therefore triangles TBF and TCE are similar. It follows that $\measuredangle TFA = \measuredangle TEA$, hence the points A, I, E, F, T lie on the same circle. It follows that $\measuredangle ATI = \measuredangle AFI = 90^\circ$ which completes our proof. □

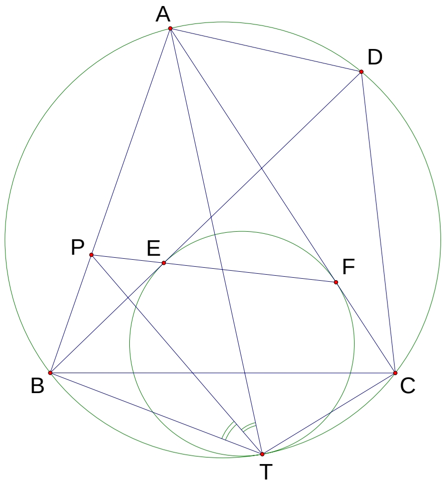
Problem 2. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let ω be a circle internally tangent to Ω at T, and to DB and AC at E and F, respectively. Let P be the intersection of EF and AB. Prove that TP is the internal angle bisector of the angle $\measuredangle ATB$.

Solution. From our lemma, applied to circles Ω, ω and points A, B, we conclude that $\frac{AT}{BT} = \frac{AF}{BE}$, thus it suffices to prove that

$$\frac{AF}{BE} = \frac{AP}{PB}.$$

Indeed, notice that $\measuredangle PEB = \measuredangle AFP$, and from the Law of Sines, applied to triangles APF, BPE, we have

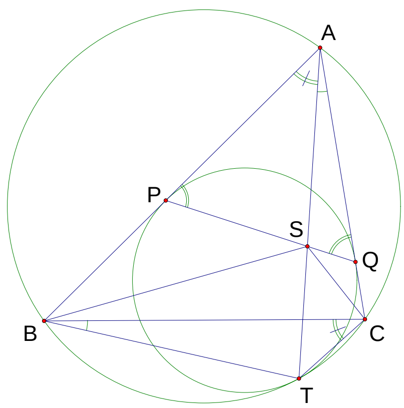
$$\frac{AP}{AF} = \frac{\sin \measuredangle AFP}{\sin \measuredangle APF} = \frac{\sin \measuredangle BEP}{\sin \measuredangle BPE} = \frac{BP}{BE}.$$



Therefore $\frac{AF}{BE} = \frac{AP}{PB}$, which completes our solution. $\square$

The third problem comes from the Moldovan Team Selection Test in 2007, which can be found in [2] and [3].

Problem 3. Let ABC be a triangle and let Ω be its circumcircle. Circles ω is internally tangent to Ω at T, and to sides AB and AC at P and Q, respectively. Let S be the intersection of AT and PQ. Prove that $\measuredangle SBA = \measuredangle SCA$.



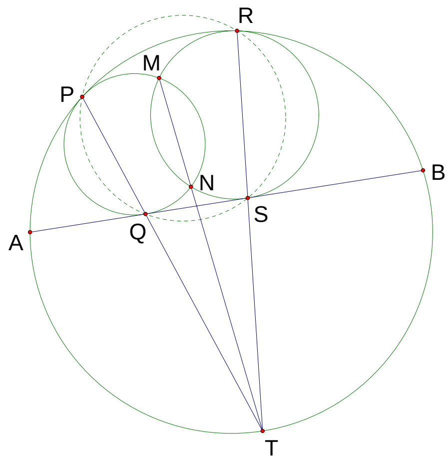
Solution. Using our lemma, we have

$$\frac{BP}{CQ} = \frac{BT}{CT} = \frac{\sin \measuredangle BCT}{\sin \measuredangle CBT} = \frac{\sin \measuredangle BAT}{\sin \measuredangle CAT} = \frac{PS}{QS}.$$

This fact implies that BPS and CQS are similar triangles which in turn implies that $\measuredangle SBA = \measuredangle SCA$. □

Problem 4. Consider a circle (O) and a chord AB. Let circles (O_1), (O_2) be internally tangent to (O) and AB and let M and N their intersection. Prove that MN passes through the midpoint of the arc AB which does not contain M and N.

Solution. Denote by P and Q the tangency points of the circle (O_1) with (O) and AB, respectively. Let R and S be the tangency points of circle (O_2) with (O) and AB, respectively. Let T be the middle point of the arc AB which does not contain M and N.



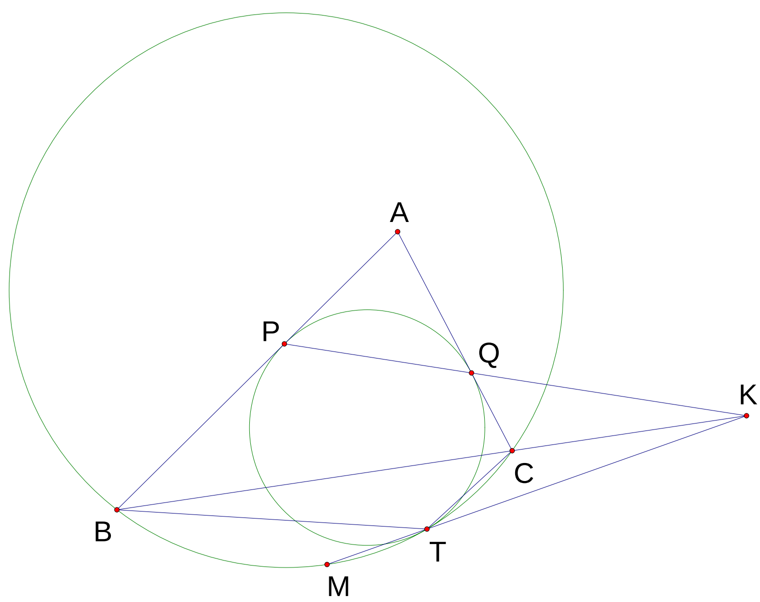
Applying the above lemma to circles (O), (O_1), and points A, B along with their tangent lines AQ, BQ to (O_1) we get $\frac{PA}{PB} = \frac{QA}{AB}$. This means that PQ passes through T. Similarly, RS passes through T. On the other hand, $\measuredangle PQA = \measuredangle QTA + \measuredangle QAT = \measuredangle PRA + \measuredangle ART = \measuredangle PRS$, therefore, points P, Q, R, S lie on a circle which we will denote by (O_3). We have that PQ is the radical axis of (O_1) and (O_3), RS is the radical axis of (O_2) and (O_3), and MN is the radical axis of (O_1) and (O_2). So, MN, PQ, and RS are concurrent at the radical center of the three circles. Hence, we deduce that MN passes through T, which is the midpoint of the arc AB that does not contain M and N. □

We continue with a problem from the MOSP Tests 2007 [4].

Problem 5. Let ABC be a triangle. Circle ω passes through points B and C. Circle ω_1 is tangent internally to ω and also to the sides AB and AC at T, P, and Q, respectively. Let M be midpoint of arc BC (containing T) of ω. Prove that lines PQ, BC, and MT are concurrent.

Solution. Let $K = PQ \cap BC$ and let $K' = MT \cap BC$. Applying Menelaos' Theorem in triangle ABC we obtain

$$\frac{KB}{KB} \cdot \frac{QC}{QA} \cdot \frac{PA}{PB} = 1 \implies \frac{KB}{KC} = \frac{BP}{CQ}.$$



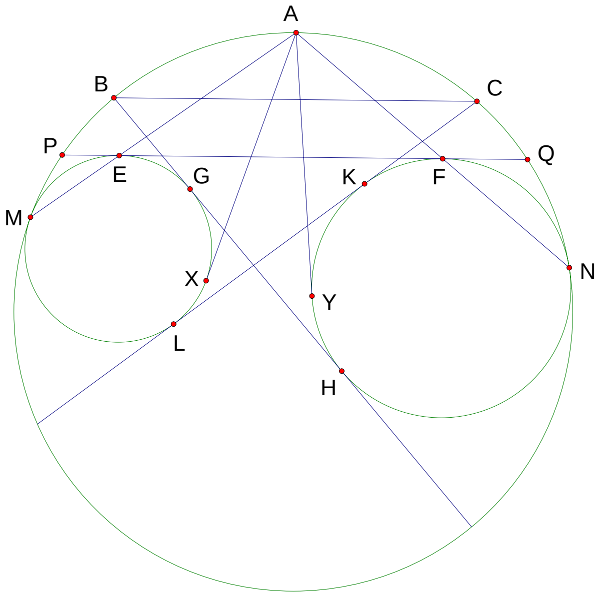
On the other hand, M is the midpoint of arc BC (containing T) of ω so MT is the external bisector of angle $\measuredangle BTC$, therefore $\frac{K'B}{K'C} = \frac{TB}{TC}$. Thus, we are left to prove that $\frac{BP}{CQ} = \frac{TB}{TC}$, which is true according to our lemma and we are done. □

The last problem was given in [5] and is also discussed and proved in [6]. Now, we will present another solution for this nice problem.

Problem 6. Circles (O_1) and (O_2) are internally tangent to a given circle (O) at M and N, respectively. Their internal common tangents intersect (O) at four points. Let B and C be two of them such that B and C lie on the same side with respect to O_1O_2. Prove that BC is parallel to an external common tangent of (O_1) and (O_2).

Solution. Draw the internal common tangents GH, KL of (O_1), (O_2) such that G and L lie on (O_1) and K and H lie on (O_2). Let EF be the external common tangent of (O_1), (O_2) such that E and B lie on the same side with respect to O_1O_2. Denote by P and Q the intersections of EF with (O). We will prove that BC is parallel to PQ. Denote by A be the midpoint of the arc PQ which does not contain M and N. Let AX and AY be the tangents at X and Y to the circles (O_1)

and (O_2). In the solution to Problem 4 we have proved that A, E, and M are collinear; A, F, and N are collinear, and the quadrilateral $MEFN$ is cyclic. Therefore, $AX^2 = AE \cdot AM = AF \cdot AN = AY^2$, i.e. $AX = AY$ (1).



Based on the lemma, $\frac{MA}{AX} = \frac{MB}{BG} = \frac{MC}{CL}$. On the other hand, by the Ptolemy's Theorem, $MA \cdot BC = MB \cdot AC = MC \cdot AB$, therefore

$$AX \cdot BC = BG \cdot AC = CL \cdot AB.$$

Similarly,

$$AY \cdot BC = BH \cdot AC + CK \cdot AB.$$

Thus $AC \cdot (BH - BG) = AB \cdot (CL - CK)$, i.e. $AC \cdot GH = AB \cdot KL$, which implies $AC = AB$. Hence, A is the midpoint of the arc BC of the circle (O). This means that BC is parallel to PQ and our solution is complete. □

References

[1] Mathlinks, *Nice geometry*,
http://www.mathlinks.ro/viewtopic.php?t=170192

[2] Mathlinks, *A circle tangent to the circumcircle and two sides*,
http://www.mathlinks.ro/viewtopic.php?t=140464

enkele lemmata om zelf proberen te vinden

Merk op dat we bij de eerste blz. van de "Lemmas in euclidean geometry"hebben dat $OA \perp PQ$.

- het isogonaal geconjugeerd punt van een punt op de omcirkel ligt op de oneindige rechte
- A, B, C en D, E, F zijn telkens 3 punten die op 1 rechte liggen waarbij $\frac{|AB|}{|BC|} = \frac{|DE|}{|EF|}$ en X, Y, Z zijn punten op resp. AD, EB, CF met $\frac{AX}{XD} = \frac{|BY|}{|EY|} = \frac{CZ}{ZF}$. Dan geldt dat X, Y, Z op 1 rechte leggen en $\frac{|AB|}{|BC|} = \frac{|XY|}{|ZY|}$.
- Het anti-steinerpunt van OI tov de centrumdriehoek is het Feuerbachpunt F.
- De homothetie die de centrumdriehoek afbeeld op de hoofddriehoek beeldt $Sp \to I \to Na$ en $N \to O \to H \to L$ af, waarbij Sp het spiekerpunt is wordt afgebeeld op het incentrum en analoog.
- In $\triangle ABC$ zijn $X, Y \in [AB], [AC]$. H, H' zijn de hoogtepunten van ABC, AXY resp. M, N zijn het midden van $[BY], [CX]$. Dan geldt dat de istomisch verwante lijn van $XY \perp HH'$ is en $//MN$.
- In een driehoek ABC construeren we inwendige punten $x, y \in [AC], [BC]$ resp zodat $|AX| = |BY|$ en zij D het midden van de boog ACB dan is $XYCD$ steeds een koordenvierhoek
- We hebben 2 lijnen a, b die snijden in een punt T en een ander vast punt P. We trekken 2 lijnen x, y door P die a, b snijden in A, B, C, D. $AC \cup BD = Q$, dan is de rechte QT steeds dezelfde (dus rechten x, y veranderenderen heeft een Q' collineair met Q, T)
- Zij $ABCD$ een koordenvierhoek en X, Y op de rechten AC, BD zodat $XY//BC$ dan is $ADXY$ cyclisch. (Reim's stelling) Merk op dat dit ook geldt voor X, Y op AB, CD met $XY//BC$ dat $ADXY$ ook cyclisch is en dit ook geldt in omgekeerde zin (2 koordenvierhoeken, rechte door snijpunten snijdt in 4 punten, waarvan er 2 bij 2 evenwijdig zijn).

3 deelverhoudingen en polen

Een deelverhouding wordt als volgt gedefinieerd; P is een punt op AB, dan is de unieke deelverhouding $(ABP) = \frac{\vec{PA}}{\vec{PB}}$.

Een dubbelverhouding is het product van 2 deelverhoudingen:

A, B, C, D zijn 4 punten op een rechte, dan is de dubbelverhouding $(ABCD) = \frac{(ABC)}{(ABD)} = \frac{\vec{CA} * \vec{DB}}{\vec{CB}\vec{DA}}$. Als $(ABCD) = k$, dan geldt dat $(ABDC) = \frac{1}{k}, (ACBD) = 1 - k$.

Een vierstraal = 4 rechten door 1 punt. Wordt een vierstraal gesneden door een rechte, is de dubbelverhouding constant.

In een cirkel geldt $\forall P$ op de cirkel en A, B, C, D vaste punten op die cirkel: (PA, PB, PC, PD) constant is voor alle P. (Hiermee bedoelen we dat een rechte die de vierstraal snijdt in 4 punten, deze verdeelt in een constante dubbelverhouding)

Gegeven een cirkel c en punt P; de middellijn van c door P snijdt de cirkel in A en B. Kiest men het punt P' op deze lijn zodat $(ABPP') = -1$, dan is de loodlijn p op de middellijn door P' de poollijn v.h. punt P tov cirkel c. P is de pool v.d. rechte p tov c.

De volgende eigenschappen gelden:

- Als Q op de poollijn van P ligt, ligt P op de poollijn van Q
- Een willekeurige lijn door P en c snijdt die cirkel in A, B, de poollijn wordt gesneden in X, dan geldt dat $(ABXP) = -1$.
- A, B, C, D liggen op een rechte in die volgorde en $(ACBD) = -1$, X ligt niet op die rechte.

  Er geldt dat $\angle BXD = 90°$ a.e.s.a. BX de bissectrice is van $\angle AXC$.

  Wanneer beide gelden, volgt ook dat $(ACBD) = -1$, vb. heeft dat $(AXII_a) = -1$ met X het snijpunt van BC met de bissectrice van $\angle BAC$.

  gevolg:De loodlijnen uit I, I_a zijn resp. X, Y en Z is het spiegelbeeld van X tov I. Dan geldt dat A, Y, Z op een rechte liggen.
- ABCD is een koordenvierhoek, E, K, J zijn de snijpunten van AC, BD,AB, CD en AD, BC, dan geldt dat E de pool is van de poollijn JK en analoog J van poollijn EK en is de poollijn van $K = EJ$. gevolg: O is de omcentrum van $ABCD$ en is het hoogtepunt van $\triangle EJK$ andere stelling hiermee: laat KE AB en CD snijden in P, Q dan geldt dat $(KEPQ) = -1$
- De pool van een lijn door2 polen ligt op het snijpunt van de 2 poollijnen (duaal/ geldt in de 2 richtingen)
- De poollijnen van 3 collineaire punten zijn concurrent (opnieuw een duale stelling)
- Als AX, BY, CZ drie concurrente lijnen zijn , waarbij X, Y, Z op BC, AC, AB resp. liggen en $T = YZ \cup BC$, dan is $(BXCT) = -1$

- DEF is de orthic triangle van $\triangle ABC$ (AD tot CF zijn hoogtelijnen) en $D' = EF \cup BC$, $E' = AC \cup FD$ en $F' = AB \cup DE$ dan geldt dat D, E, F op een rechte liggen.

Stelling 3.1. *(harmonische vierhoeken)*

Binnen een koordenvierhoek $PQRS$ zijn volgende eigenschappen equivalent:

1. *$PQRS$ is harmonisch*
2. *$|PQ||RS| = |PS||RQ|$*
3. *PR is de P-symmedian of $\triangle QPS$*
4. *QS is de Q-symmedian van $\triangle PQR$*
5. *de raaklijnen in P, R aan de omgeschreven cirkel snijden op QS*
6. *de raaklijnen in Q, S aan de omgeschreven cirkel snijden op PR*
7. *TA, TB, TC, TD of $T(abcd)$ is een harmonische vierstraal waarbij T een ander punt is op de omgeschreven cirkel*

De volgende PDF geeft reeds vele voorbeelden om de kracht te tonen.

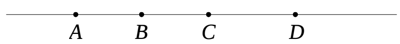
Harmonic Division and its Applications

Cosmin Pohoata

Let d be a line and A, B, C and D four points which lie in this order on it. The four-point $(ABCD)$ is called a *harmonic division*, or simply *harmonic*, if (using directed lengths)

$$\frac{CA}{CB} = -\frac{DA}{DB}.$$

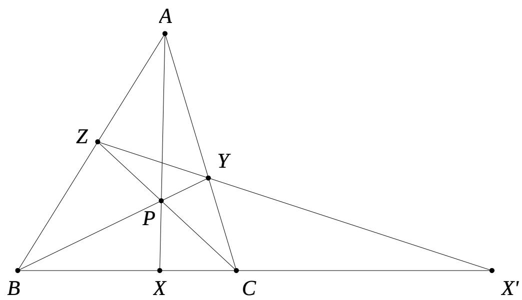
If X is a point not lying on d, then we say that pencil $X(ABCD)$ (which consists of the four lines XA, XB, XC, XD) is harmonic if $(ABCD)$ is harmonic.



In this note, we show how to use harmonic division as a tool in solving some difficult Euclidean geometry problems.

We begin by stating two very useful lemmas without proof. The first lemma shows one of the simplest geometric characterizations of harmonic divisions, based on the theorems of Menelaus and Ceva.

Lemma 1. *In a triangle ABC consider three points X, Y, Z on the sides BC, CA, respective AB. If X' is the point of intersection of YZ with the extended side BC, then the four-point $(BXCX')$ forms and harmonic division if and only if the cevians AX, BY and CZ are concurrent.*



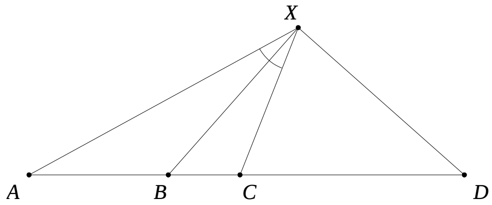
The second lemma is a consequence of the Appollonius circle property. It can be found in [1] followed by several interpretations.

Lemma 2. *Let four points A, B, C and D, in this order, lying on d. Then, if two of the following three propositions are true, then the third is also true:*

(1) The division $(ABCD)$ is harmonic.

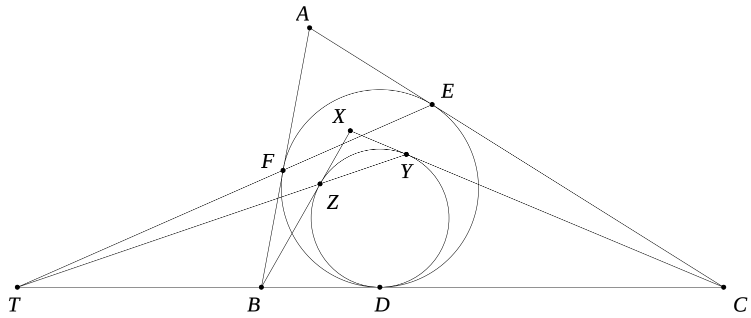
(2) XB is the internal angle bisector of $\angle AXC$.

(3) $XB \perp XD$.



We begin our journey with a problem from the IMO 1995 Shortlist.

Problem 1. Let ABC be a triangle, and let D, E, F be the points of tangency of the incircle of triangle ABC with the sides BC, CA and AB respectively. Let X be in the interior of ABC such that the incircle of XBC touches XB, XC and BC in Z, Y and D respectively. Prove that $EFZY$ is cyclic.

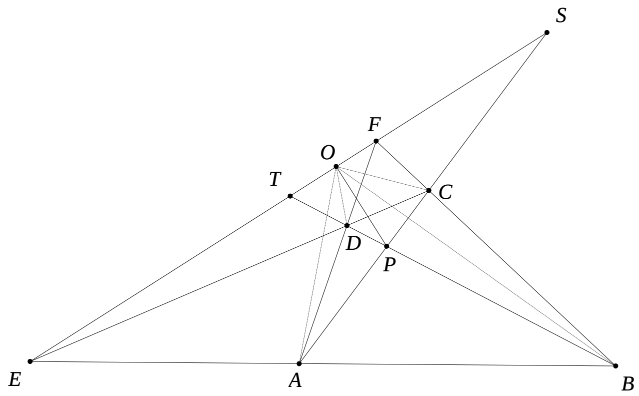


Solution. Denote $T = BC \cap EF$. Because of the concurency of the lines AD, BE, CF in the Gergonne point of triangle ABC, we deduce that the division $(TBDC)$ is harmonic. Similarly, the lines XD, BY and CZ are concurrent in the Gergonne point of triangle XBC, so $T \in YZ$ as a consequence of Lemma 1.

Now expressing the power of point T with respect to the incircle of triangle ABC and the incircle of triangle XBC we have that $TD^2 = TE \cdot TF$ and $TD^2 = TZ \cdot TY$. So $TE \cdot TF = TZ \cdot TY$, therefore the quadrilateral $EFZY$ is cyclic. $\square$

For our next application, we present a problem given at the Chinese IMO Team Selection Test in 2002.

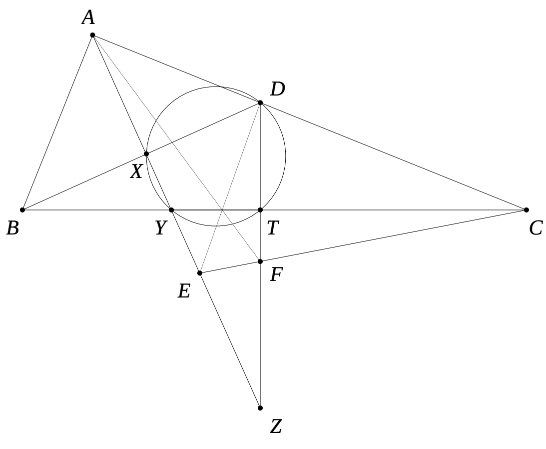
Problem 2. Let $ABCD$ be a convex quadrilateral. Let $E = AB \cap CD$, $F = AD \cap BC$, $P = AC \cap BD$, and let O the foot of the perpendicular from P to the line EF. Prove that $\angle BOC = \angle AOD$.



Solution. Denote $S = AC \cap EF$ and $T = BD \cap EF$. As from Lemma 1, we deduce that the division $(ETFS)$ is harmonic. Furthermore, the division $(APCS)$ is also harmonic, due to the pencil $B(ETFS)$. But now, the pencil $E(APCS)$ is harmonic, so by intersecting it with the line BD, it follows that the four-point $(BPDT)$ is harmonic. Therefore, the pencil $O(APCS)$ is harmonic and $OP \perp OS$, thus by Lemma 2, $\angle POA = \angle POC$. Similarly, the pencil $O(BPDT)$ is harmonic and $OP \perp OT$, thus again by Lemma 2, $\angle POB = \angle POD$. It follows that $\angle AOD = \angle BOC$. □

We continue with an interesting problem proposed by Dinu Serbanescu at the Romanian Junior Balkan MO 2007, Team Selection Test.

Problem 3. Let ABC be a right triangle with $\angle A = 90°$ and let D be a point on side AC. Denote by E the reflection of A across the line BD and F the intersection point of CE with the perpendicular to BC at D . Prove that AF, DE and BC are concurrent.



Solution. Denote the points $X = AE \cap BD$, $Y = AE \cap BC$, $Z = AE \cap DF$ and $T = DF \cap BC$. From Lemma 1, applied to triangle AEC and for the cevians AF

and ED, we observe that the lines AF, DE and BC are concurrent if and only if the division $(AYEZ)$ is harmonic.

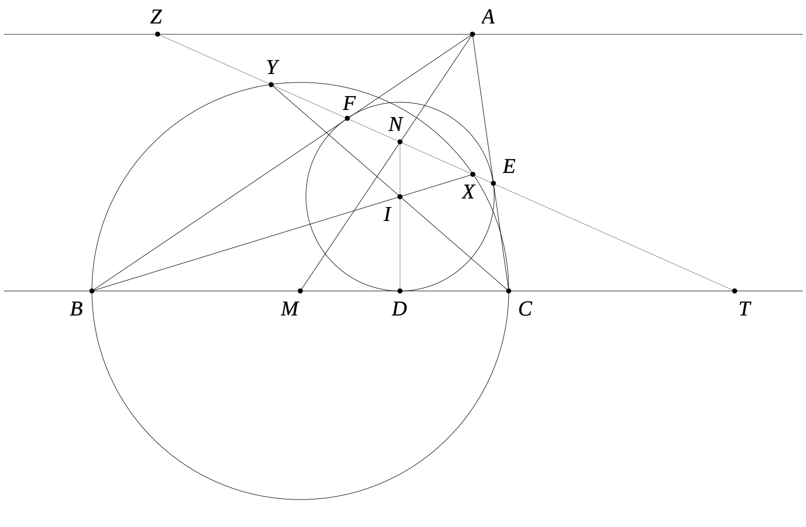
Since the quadrilateral $XYTD$ is cyclic, $\tan XYB = \tan XDZ$, which is equivalent to $XB/XY = XZ/XD$. So $XB \cdot XD = XY \cdot XZ$.

Since triangles XAB and XDA are similar, we have that $XA^2 = XB \cdot XD$, so $XA^2 = XY \cdot XZ$. Using $XA = XE$, we obtain that $\frac{YA}{YE} = \frac{ZA}{ZE}$, and thus the division $(AYEZ)$ is harmonic. □

The next problem was proposed by the author and given at the Romanian IMO Team Selection Test in 2007.

Problem 4. Let ABC be a triangle, let E, F be the tangency points of the incircle $\Gamma(I)$ to the sides AC, respectively AB, and let M be the midpoint of the side BC. Let $N = AM \cap EF$, let $\gamma(M)$ be the circle of diameter BC, and let lines BI and CI meet γ again at X and Y, respectively. Prove that

$$\frac{NX}{NY} = \frac{AC}{AB}.$$



Solution. We will assume $AB \geq AC$, so the solution matches a possible drawing. Let $T = EF \cap BC$ (for $AB = AC$, $T = \infty$), and D the tangency point of Γ to BC.

Claim 1. In the configuration described above, for $X' = BI \cap EF$, one has $BX' \perp CX'$.

Proof. The fact that BI effectively intersects EF follows from $\angle DFE = \frac{1}{2}(\angle ABC + \angle BAC) = \frac{1}{2}\pi - \frac{1}{2}\angle ACB < \frac{1}{2}\pi$, and $BI \perp DF$ (similarly, CI effectively intersects EF).

The division $(TBDC)$ is harmonic, and triangles BFX' and BDX' are congruent, therefore $\angle TX'B = \angle DX'B$, which is equivalent to $BX' \perp CX'$ (similarly, for $Y' = CI \cap EF$, one has $CY' \perp BY'$). □

Claim 2. In the configuration described above, one has $N = DI \cap EF$.

Proof. It is enough to prove that $NI \perp BC$. Let d be the line through A, parallel to BC. Since the pencil $A(BMC\infty)$ is harmonic, it follows the division $(FNEZ)$ is harmonic, where $Z = d \cap EF$. Therefore N lies on the polar of Z relative to circle Γ, and as $N \in EF$ (the polar of A), it follows that AZ is the polar of N relative to circle Γ, hence $NI \perp d$, so $NI \perp BC$. In conclusion, since $DI \perp BC$, one has $N \in DI$. □

It follows, according to Claim 1, that $X = X'$ and $Y = Y'$, therefore $X, Y \in EF$. Since the division $(TBDC)$ is harmonic, it follows that D lies on the polar p of T relative to circle γ. But $TM \perp p$, so $BC \perp p$, and since $DI \perp BC$, it follows that p is, in fact, DI.

Now, according to Claim 2, it follows that D, I, N are collinear. Since DN is the polar, it means the division $(TYNX)$ is harmonic, thus the pencil $D(TYNX)$ is harmonic. But $DT \perp DN$, so DN is the angle bisector of $\angle XDY$, hence

$$\frac{NX}{NY} = \frac{DX}{DY} = \frac{\sin \angle DYX}{\sin \angle DXY}.$$

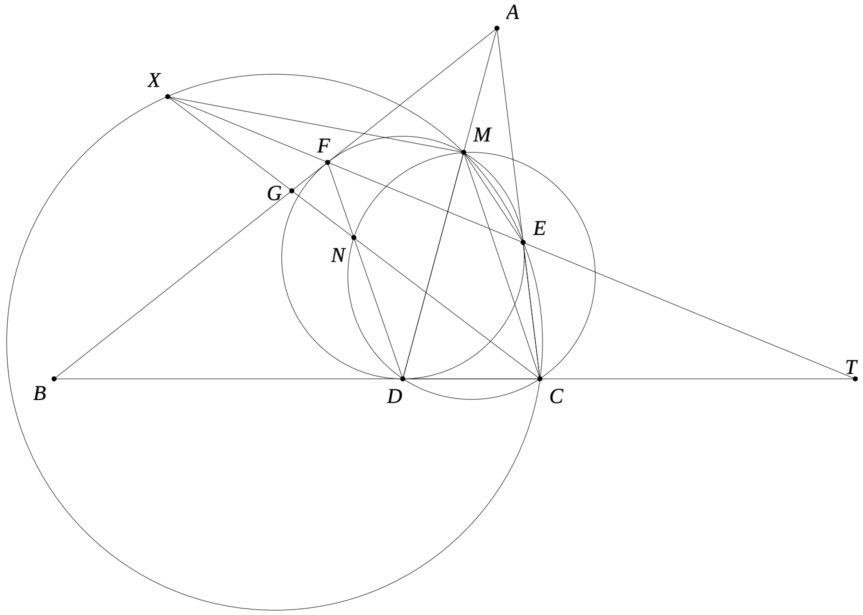
As quadrilaterals $BDIY$ and $CDIX$ are cyclic (since pairs of opposing angles are right angles), it follows that $\frac{1}{2}\angle ABC = \angle DBI = \angle DYI = \frac{1}{2}\angle DYX$ (triangles CDY and CEY are congruent), so $\angle DYX = \angle ABC$. Similarly, $\angle DXY = \angle ACB$. Therefore

$$\frac{NX}{NY} = \frac{DX}{DY} = \frac{\sin \angle DYX}{\sin \angle DXY} = \frac{\sin \angle ABC}{\sin \angle ACB} = \frac{AC}{AB}.$$

□

The following problem was posted on the MathLinks forum [2]:

Problem 5. Let ABC be a triangle and $\rho(I)$ its incircle. D, E and F are the points of tangency of $\rho(I)$ with BC, CA and AB respectively. Denote $M = \rho(I) \cap AD$, N the intersection of the circumcircle of CDM with DF and $G = CN \cap AB$. Prove that $CD = 3FG$.



Solution. Denote $X = EF \cap CG$ and $T = EF \cap BC$. Now because the four-point $(TBDC)$ forms an harmonic division, so does the pencil $F(TBDC)$ and now by intersecting it with the line CG, we obtain that the division $(XGNC)$ is harmonic.

According to the Menelaus theorem applied to BCG for the transversal DNF, we find that $CD = 3GF$ is equivalent to $CN = 3NG$.

Since $(XGNC)$ is harmonic, $\frac{NC}{NG} = \frac{XC}{XG}$, so it suffices to show that N is the midpoint of CX.

Observe that $\angle MEX = \angle MDF = \angle MCX$, therefore the quadrilateral $MECX$ is cyclic, which implies that $\angle MXC = \angle MEA = \angle ADE$ and $\angle MCX = \angle ADF$.

Also, $\angle CMN = \angle FDB$ and $\angle XMN = \angle XMC - \angle CMN = \angle CEF - \angle FDB = \angle EDC$.

Using the above angle relations and the equation

$$\frac{NX}{NC} = \frac{\sin \angle MCX}{\sin \angle MXC} \cdot \frac{\sin \angle XMN}{\sin \angle CMN},$$

we obtain that $NC = NX$, so

$$\frac{\sin \angle FDA}{\sin \angle EDA} = \frac{\sin \angle BDF}{\sin \angle CDE}.$$

On other hand, DA coincides with a symmedian of triangle DEF, so

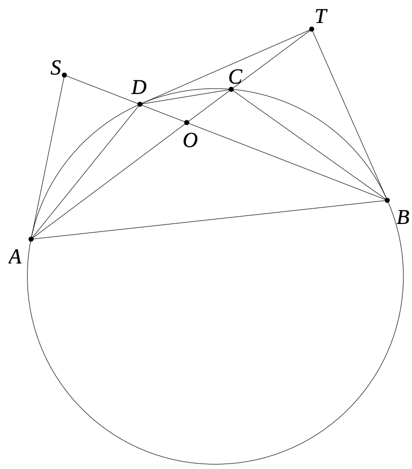
$$\frac{\sin \angle FDA}{\sin \angle EDA} = \frac{FD}{ED} = \frac{\sin \angle DEF}{\sin \angle DFE} = \frac{\sin \angle BDF}{\sin \angle CDE}.$$

Therefore, N is the midpoint of CX. □

Let $ABCD$ be a cyclic quadrilateral and X a point on the circle. Then, the $ABCD$ is called *harmonic* if the pencil $X(ABCD)$ is harmonic. For a list of properties regarding the harmonic quadrilateral, interested readers may can consult [1] and [3].

The following problem was given at an IMO Team Preparation Contest, held in Bacau, Romania, in 2006.

Problem 6. Let $ABCD$ be a convex quadrilateral, for which denote $O = AC \cap BD$. If BO is a symmedian of triangle ABC and DO is a symmedian of triangle ADC, prove that AO is a symmedian of triangle ABD.



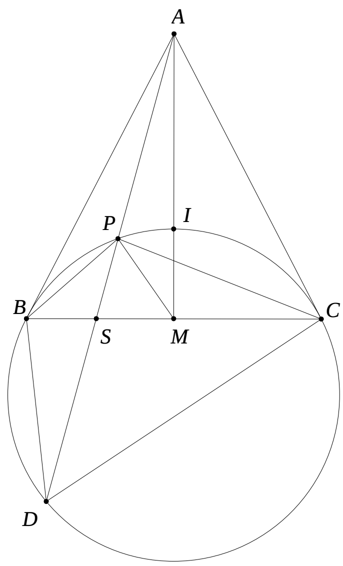
Solution. Denote $T_1 = DD \cap AC$, $T_2 = BB \cap AC$, $T = BB \cap DD$, where DD, respective BB represents the tangent in D to the circumcircle of ADC and the tangent in B to the circumcircle of ABC.

Since BO is a symmedian of triangle ABC and DO is a symmedian of triangle ADC, the divisions $(AOCT_1)$ and $(AOCT_2)$ are harmonic, so $T_1 = T_2 = T$.

Hence, BD is the polar of T_1 with respect to the circumcircle of ADC and also the polar of T_2 with respect to the circumcircle of ABC. But because $T_1 = T_2$, we deduce that the circles ABC and ADC coincide, i.e. the quadrilateral $ABCD$ is cyclic, and since the division $(AOCT)$ is harmonic, the pencil $D(AOCT)$ is, and by intersecting it by the circle $ABCD$, it follows that the quadrilateral $ABCD$ is also harmonic. Then, the pencil $A(ABCD)$ is harmonic. By intersecting it with the line BD, we see that the division $(BODS)$ is harmonic, where $S = AA \cap BD$. It follows that AO is a symmedian of triangle BAD. □

The next problem was also given in an IMO Team Preparation Test, at the IMAR Contest, held in Bucharest in 2006.

Problem 7. Let ABC be an isosceles triangle with $AB = AC$, and M the midpoint of BC. Find the locus of the point P interior to the triangle for which $\angle BPM + \angle CPA = \pi$.



Solution. Denote the point D as the intersection of the line AP with the circumcircle of BPC and $S = DP \cap BC$.

Since $\angle SPC = 180 - \angle CPA$, it follows that $\angle BPS = \angle CPM$.

From the Steiner theorem applied in to triangle BPC for the isogonals PS and PM,

$$\frac{SB}{SC} = \frac{PB^2}{PC^2}.$$

On other hand, using Sine Law, we obtain

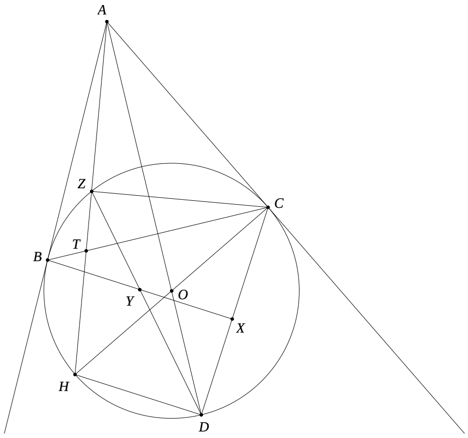
$$\frac{SB}{SC} = \frac{DB}{DC} \cdot \frac{\sin \angle SDB}{\sin \angle SDC} = \frac{DB}{DC} \cdot \frac{\sin \angle PCB}{\sin \angle PBC} = \frac{DB}{DC} \cdot \frac{PB}{PC}.$$

Thus by the above relations, it follows that $\frac{DB}{DC} = \frac{PB}{PC}$, i.e. the quadrilateral $PBDC$ is harmonic, therefore the point $A' = BB \cap CC$ lies on the line PD.

If $A' = A$, then lines AB and AC are always tangent to the circle BPC, and so the locus of P is the circle BIC, where I is the incircle of ABC. Otherwise, if $A' \neq A$, then $A' = AM \cap PS \cap BB \cap CC$, due to the fact that $A' \in PD$ and and $A = PS \cap AM$, therefore by maintaining the condition that $A' \neq A$, we obtain that $PS = AM$, therefore P lies on AM. □

The next problem was selected in the Senior BMO 2007 Shortlist, proposed by the author.

Problem 8. Let $\rho(O)$ be a circle and A a point outside it. Denote by B, C the points where the tangents from A with respect to $\rho(O)$ meet the circle, D the point on $\rho(O)$, for which $O \in AD$, X the foot of the perpendicular from B to CD, Y the midpoint of the line segment BX and by Z the second intersection of DY with $\rho(O)$. Prove that $ZA \perp ZC$.

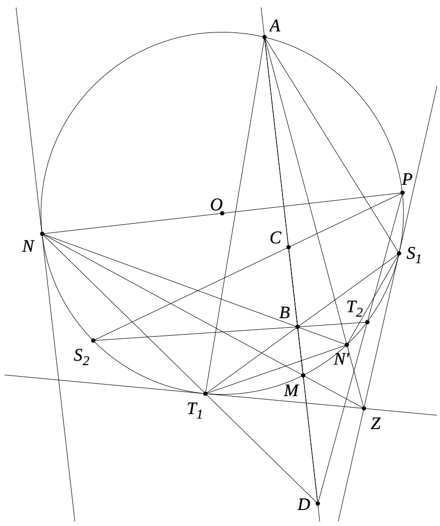


Solution. Let us call $H = CO \cap \rho(O)$. Thus $DC \perp DH$, so $DH \| BX$.

Because Y is the midpoint of BX, we deduce that the division $(BYX\infty)$ is harmonic, so also is the pencil $D(BYXH)$ and by intersecting it with $\rho(O)$, it follows that the quadrilateral $HBZC$ is harmonic. Then, the pencil $C(HBZC)$ is harmonic, so by intersecting it with the line HZ, it follows that the division $(A'ZTH)$ is harmonic, where $A' = HZ \cap CC$ and $T = HZ \cap BC$.

So, the line CH is the polar of A' with respect to $\rho(O)$, but $CH = BC$ is the polar of A as well, so $A = A'$, hence the points H, Z, A are collinear, therefore $ZA \perp ZC$. □

The last problem is a generalization of a problem by Virgil Nicula [4]. The solution covers all concepts and methods presented throughout this paper.



Problem 9. Let d be a line and A, C, B, D four points in this order on it such that the division $(ACBD)$ is harmonic. Denote by M the midpoint of the line segment CD. Let ω be a circle passing through A and M. Let NP be the diameter of ω perpendicular to AM. Let lines NC, ND, PC, PD meet ω again at S_1, T_1, S_2, T_2, respectively. Prove that $B = S_1T_1 \cap S_2T_2$.

Solution. Since the four-point $(ACBD)$ is harmonic, so is the pencil $N(ACBD)$ and by intersecting it with ω, it follows that the quadrilateral $AS_1N'T_1$ is harmonic, hence the lines S_1S_1, T_1T_1 and AN' are concurrent, where $N' = NB \cap \omega$.

Because the tangent in N to ω is parallel with the line AM and since M is the midpoint of CD, the division $(CMD\infty)$ is harmonic, therefore the pencil $N(NDMC)$ also is, and by intersecting it with ω, it follows that the quadrilateral NT_1MS_1 is harmonic, hence the lines S_1S_1, T_1T_1 and MN are concurrent.

From the above two observations, we deduce that the lines S_1S_1, T_1T_1, MN, AN' are concurrent at a point Z.

On the other hand, since the pencils $B(AS_1N'T_1)$ and $B(NT_1MS_1)$ are harmonic, by intersecting them with ω, it follows that the quadrilaterals NT_3MS_3 and $AS_3N'T_3$ harmonic, where $S_3 = BS_1 \cap \omega$ and $T_3 = BT_1 \cap \omega$.

Similarly, we deduce that the lines S_3S_3, T_3T_3, MN and AN' are concurrent in the same point Z.

Therefore, S_3T_3 is the polar of Z with respect to ω, but so is S_1T_1, thus $S_1T_1 = S_3T_3$, so $S_1 = S_3$ and $T_1 = T_3$, therefore the points S_1, B, T_1 are collinear.

Similarly, the points S_2, B, T_2 are collinear, from which it follows that $B = S_1T_1 \cap S_2T_2$. □

References

[1] Virgil Nicula, Cosmin Pohoata. *Diviziunea armonica.* GIL, 2007.

[2] `http://www.mathlinks.ro/viewtopic.php?t=151320`

[3] `http://www.mathlinks.ro/viewtopic.php?t=70184`

[4] `http://www.mathlinks.ro/viewtopic.php?t=155875`

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Casey's Theorem and its Applications

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Abstract. We present a proof of the generalized Ptolemy's theorem, also known as Casey's theorem and its applications in the resolution of difficult geometry problems.

1 Casey's Theorem.

Theorem 1. Two circles $\Gamma_1(r_1)$ and $\Gamma_2(r_2)$ are internally/externally tangent to a circle $\Gamma(R)$ through A, B, respetively. The length δ_{12} of the common external tangent of Γ_1, Γ_2 is given by:

$$\delta_{12} = \frac{AB}{R}\sqrt{(R \pm r_1)(R \pm r_2)}$$

Proof. Without loss of generality assume that $r_1 \geq r_2$ and we suppose that Γ_1 and Γ_2 are internally tangent to Γ. The remaining case will be treated analogously. A common external tangent between Γ_1 and Γ_2 touches Γ_1, Γ_2 at A_1, B_1 and A_2 is the orthogonal projection of O_2 onto O_1A_1. (See Figure 1). By Pythagorean theorem for $\triangle O_1O_2A_2$, we obtain

$${\delta_{12}}^2 = (A_1B_1)^2 = (O_1O_2)^2 - (r_1 - r_2)^2$$

Let $\angle O_1OO_2 = \lambda$. By cosine law for $\triangle OO_1O_2$, we get

$$(O_1O_2)^2 = (R - r_1)^2 + (R - r_2)^2 - 2(R - r_1)(R - r_2)\cos\lambda$$

By cosine law for the isosceles triangle $\triangle OAB$, we get

$$AB^2 = 2R^2(1 - \cos\lambda)$$

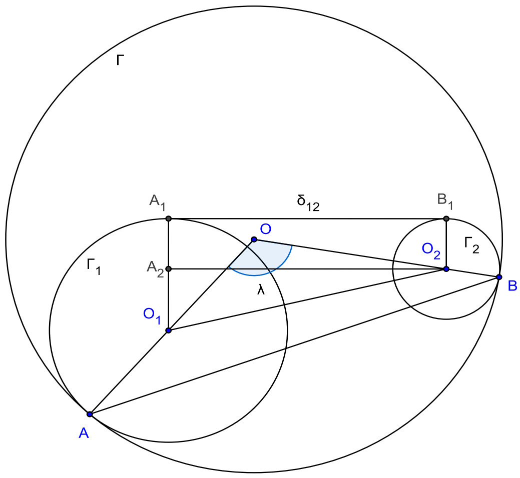

Figure 1: Theorem 1

Eliminating $\cos\lambda$ and O_1O_2 from the three previous expressions yields

$$\delta_{12}{}^2 = (R-r_1)^2 + (R-r_2)^2 - (r_1-r_2)^2 - 2(R-r_1)(R-r_2)\left(1 - \frac{AB^2}{2R^2}\right)$$

Subsequent simplifications give

$$\delta_{12} = \frac{AB}{R}\sqrt{(R-r_1)(R-r_2)} \quad (1)$$

Analogously, if Γ_1, Γ_2 are externally tangent to Γ, then we will get

$$\delta_{12} = \frac{AB}{R}\sqrt{(R+r_1)(R+r_2)} \quad (2)$$

If Γ_1 is externally tangent to Γ and Γ_2 is internally tangent to Γ, then a similar reasoning gives that the length of the common internal tangent between Γ_1 and Γ_2 is given by

$$\delta_{12} = \frac{AB}{R}\sqrt{(R+r_1)(R-r_2)} \quad (3)$$

Theorem 2 (Casey). Given four circles $\Gamma_i, i = 1, 2, 3, 4$, let δ_{ij} denote the length of a common tangent (either internal or external) between Γ_i and Γ_j. The four circles are tangent to a fith circle Γ (or line) if and only if for appropriate choice of signs,

$$\delta_{12} \cdot \delta_{34} \pm \delta_{13} \cdot \delta_{42} \pm \delta_{14} \cdot \delta_{23} = 0$$

The proof of the direct theorem is straightforward using Ptolemy's theorem for the quadrilateral $ABCD$ whose vertices are the tangency points of $\Gamma_1(r_1), \Gamma_2(r_2), \Gamma_3(r_3), \Gamma_4(r_4)$ with $\Gamma(R)$. We susbtitute the lengths of its sides and digonals in terms of the lenghts of the tangents δ_{ij}, by using the formulas (1), (2) and (3). For instance, assuming that all tangencies are external, then using (1), we get

$$\delta_{12} \cdot \delta_{34} + \delta_{14} \cdot \delta_{23} = \left(\frac{AB \cdot CD + AD \cdot BC}{R^2}\right) \sqrt{(R - r_1)(R - r_2)(R - r_3)(R - r_4)}$$

$$\delta_{12} \cdot \delta_{34} + \delta_{14} \cdot \delta_{23} = \left(\frac{AC \cdot BD}{R^2}\right) \sqrt{(R - r_1)(R - r_3)} \cdot \sqrt{(R - r_2)(R - r_4)}$$

$$\delta_{12} \cdot \delta_{34} + \delta_{14} \cdot \delta_{23} = \delta_{13} \cdot \delta_{42}.$$

Casey established that this latter relation is sufficient condition for the existence of a fith circle $\Gamma(R)$ tangent to $\Gamma_1(r_1), \Gamma_2(r_2), \Gamma_3(r_3), \Gamma_4(r_4)$. Interestingly, the proof of this converse is a much tougher exercise. For a proof you may see [1].

2 Some Applications.

I) $\triangle ABC$ is isosceles with legs $AB = AC = L$. A circle ω is tangent to $\overline{BC}$ and the arc BC of the circumcircle of $\triangle ABC$. A tangent line from A to ω touches ω at P. Describe the locus of P as ω varies.

Solution. We use Casey's theorem for the circles $(A), (B), (C)$ (with zero radii) and ω, all internally tangent to the circumcircle of $\triangle ABC$. Thus, if ω touches $\overline{BC}$ at Q, we have:

$$L \cdot CQ + L \cdot BQ = AP \cdot BC \Longrightarrow AP = \frac{L(BQ + CQ)}{BC} = L$$

The length AP is constant, i.e. Locus of P is the circle with center A and radius $AB = AC = L$.

II) (O) is a circle with diameter $\overline{AB}$ and P, Q are two points on (O) lying on different sides of $\overline{AB}$. T is the orthogonal projection of Q onto $\overline{AB}$. Let $(O_1), (O_2)$ be the circles with diameters TA, TB and PC, PD are the tangent segments from P to $(O_1), (O_2)$, respectively. Show that $PC + PD = PQ$. [2].

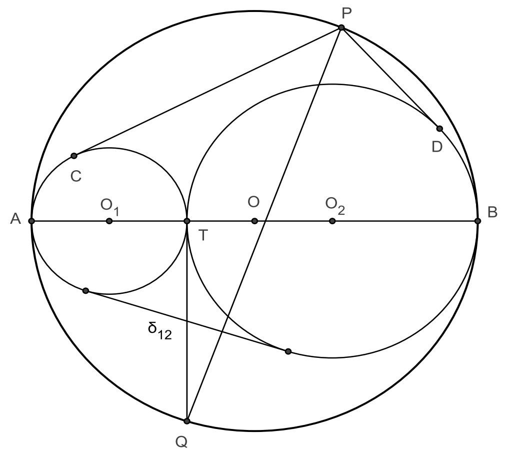

Figure 2: Application II

Solution. Let δ_{12} denote the length of the common external tangent of $(O_1), (O_2)$. We use Casey's theorem for the circles $(O_1), (O_2), (P), (Q)$, all internally tangent to (O).

$$PC \cdot QT + PD \cdot QT = PQ \cdot \delta_{12} \Longrightarrow PC + PD = PQ \cdot \frac{\delta_{12}}{QT} = PQ \cdot \frac{\sqrt{TA \cdot TB}}{TQ} = PQ.$$

III) In $\triangle ABC$, let $\omega_A, \omega_B, \omega_C$ be the circles tangent to BC, CA, AB through their midpoints and the arcs BC, CA, AB of its circumcircle (not containing A, B, C). If $\delta_{BC}, \delta_{CA}, \delta_{AB}$ denote the lengths of the common external tangents between $(\omega_B, \omega_C), (\omega_C, \omega_A)$ and (ω_A, ω_B), respectively, then prove that

$$\delta_{BC} = \delta_{CA} = \delta_{AB} = \frac{a+b+c}{4}$$

Solution. Let $\delta_A, \delta_B, \delta_C$ denote the lengths of the tangents from A, B, C to $\omega_A, \omega_B, \omega_C$, respectively. By Casey's theorem for the circles $(A), (B), (C), \omega_B$, all tangent to the circumcircle of $\triangle ABC$, we get

$$\delta_B \cdot b = a \cdot AE + c \cdot CE \Longrightarrow \delta_B = \tfrac{1}{2}(a+c)$$

Similarly, by Casey's theorem for $(A), (B), (C), \omega_C$ we'll get $\delta_C = \frac{1}{2}(a+b)$

Now, by Casey's theorem for $(B), (C), \omega_B, \omega_C$, we get $\delta_B \cdot \delta_C = \delta_{BC} \cdot a + BF \cdot BE \Longrightarrow$

$$\delta_{BC} = \frac{\delta_B \cdot \delta_C - BF \cdot BE}{a} = \frac{(a+c)(a+b) - bc}{4a} = \frac{a+b+c}{4}$$

By similar reasoning, we'll have $\delta_{CA} = \delta_{AB} = \frac{1}{4}(a+b+c)$.

IV) A circle $\mathcal{K}$ passes through the vertices B, C of $\triangle ABC$ and another circle ω touches $AB, AC, \mathcal{K}$ at P, Q, T, respectively. If M is the midpoint of the arc BTC of $\mathcal{K}$, show that BC, PQ, MT concur. [3]

Solution. Let R, ϱ be the radii of $\mathcal{K}$ and ω, respectively. Using formula (1) of Theorem 1 for $\omega, (B)$ and $\omega, (C)$. Both $(B), (C)$ with zero radii and tangent to $\mathcal{K}$ through B, C, we obtain:

$$TC^2 = \frac{CQ^2 \cdot R^2}{(R-\varrho)(R-0)} = \frac{CQ^2 \cdot R}{R-\varrho} \ , \ TB^2 = \frac{BP^2 \cdot R^2}{(R-\varrho)(R-0)} = \frac{BP^2 \cdot R}{R-\varrho} \Longrightarrow \frac{TB}{TC} = \frac{BP}{CQ}$$

Let PQ cut BC at U. By Menelaus' theorem for $\triangle ABC$ cut by $\overline{UPQ}$ we have

$$\frac{UB}{UC} = \frac{BP}{AP} \cdot \frac{AQ}{CQ} = \frac{BP}{CQ} = \frac{TB}{TC}$$

Thus, by angle bisector theorem, U is the foot of the T-external bisector TM of $\triangle BTC$.

V) If D, E, F denote the midpoints of the sides BC, CA, AB of $\triangle ABC$. Show that the incircle (I) of $\triangle ABC$ is tangent to $\odot(DEF)$. (Feuerbach theorem).

Solution. We consider the circles $(D), (E), (F)$ with zero radii and (I). The notation δ_{XY} stands for the length of the external tangent between the circles $(X), (Y)$, then

$$\delta_{DE} = \frac{c}{2} \ , \ \delta_{EF} = \frac{a}{2} \ , \ \delta_{FD} = \frac{b}{2} \ , \ \delta_{DI} = \left|\frac{b-c}{2}\right| \ , \ \delta_{EI} = \left|\frac{a-c}{2}\right| \ , \ \delta_{FI} = \left|\frac{b-a}{2}\right|$$

For the sake of applying the converse of Casey's theorem, we shall verify if, for some combination of signs + and −, we get $\pm c(b-a) \pm a(b-c) \pm b(a-c) = 0$, which is trivial. Therefore, there exists a circle tangent to $(D), (E), (F)$ and (I), i.e. (I) is internally tangent to $\odot(DEF)$. We use the same reasoning to show that $\odot(DEF)$ is tangent to the three excircles of $\triangle ABC$.

VI) $\triangle ABC$ is scalene and D, E, F are the midpoints of BC, CA, AB. The incircle (I) and 9 point circle $\odot(DEF)$ of $\triangle ABC$ are internally tangent through the Feuerbach point F_e. Show that one of the segments $\overline{F_eD}, \overline{F_eE}, \overline{F_eF}$ equals the sum of the other two. [4]

Solution. WLOG assume that $b \geq a \geq c$. Incircle (I, r) touches BC at M. Using formula (1) of Theorem 1 for (I) and (D) (with zero radius) tangent to the 9-point circle $(N, \frac{R}{2})$, we have:

$$F_eD^2 = \frac{DM^2 \cdot (\frac{R}{2})^2}{(\frac{R}{2} - r)(\frac{R}{2} - 0)} \Longrightarrow F_eD = \sqrt{\frac{R}{R - 2r}} \cdot \frac{(b - c)}{2}$$

By similar reasoning, we have the expressions

$$F_eE = \sqrt{\frac{R}{R - 2r}} \cdot \frac{(a - c)}{2} \ , \ F_eF = \sqrt{\frac{R}{R - 2r}} \cdot \frac{(b - a)}{2}$$

Therefore, the addition of the latter expressions gives

$$F_eE + F_eF = \sqrt{\frac{R}{R - 2r}} \cdot \frac{b - c}{2} = F_eD$$

VII) $\triangle ABC$ is a triangle with $AC > AB$. A circle ω_A is internally tangent to its circumcircle ω and AB, AC. S is the midpoint of the arc BC of ω, which does not contain A and ST is the tangent segment from S to ω_A. Prove that

$$\frac{ST}{SA} = \frac{AC - AB}{AC + AB} \quad [5]$$

Solution. Let M, N be the tangency points of ω_A with AC, AB. By Casey's theorem for $\omega_A, (B), (C), (S)$, all tangent to the circumcircle ω, we get

$$ST \cdot BC + CS \cdot BN = CM \cdot BS \Longrightarrow ST \cdot BC = CS(CM - BN)$$

If U is the reflection of B across AS, then $CM - BN = UC = AC - AB$. Hence

$$ST \cdot BC = CS(AC - AB) \ (\star)$$

By Ptolemy's theorem for $ABSC$, we get $SA \cdot BC = CS(AB + AC)$. Together with $(\star)$, we obtain

$$\frac{ST}{SA} = \frac{AC - AB}{AC + AB}$$

VIII) Two congruent circles $(S_1),(S_2)$ meet at two points. A line ℓ cuts (S_2) at A,C and (S_1) at B,D (A,B,C,D are collinear in this order). Two distinct circles ω_1,ω_2 touch the line ℓ and the circles $(S_1),(S_2)$ externally and internally respectively. If ω_1,ω_2 are externally tangent, show that $AB=CD$. [6]

Solution. Let $P \equiv \omega_1 \cap \omega_2$ and M,N be the tangency points of ω_1 and ω_2 with an external tangent. Inversion with center P and power $PB \cdot PD$ takes (S_1) and the line ℓ into themselves. The circles ω_1 and ω_2 go to two parallel lines k_1 and k_2 tangent to (S_1) and the circle (S_2) goes to another circle (S_2') tangent to k_1,k_2. Hence, (S_2) is congruent to its inverse (S_2'). Further, $(S_2),(S_2')$ are symmetrical about $P \Longrightarrow PC \cdot PA = PB \cdot PD$.

By Casey's theorem for $\omega_1,\omega_2,(D),(B),(S_1)$ and $\omega_1,\omega_2,(A),(C),(S_2)$ we get:

$$DB = \frac{2PB \cdot PD}{MN} \ , \ AC = \frac{2PA \cdot PC}{MN}$$

Since $PC \cdot PA = PB \cdot PD \Longrightarrow \ AC = BD \ \Longrightarrow AB = CD.$

IX) $\triangle ABC$ is equilateral with side length L. Let (O,r) and (O,R) be the incircle and circumcircle of $\triangle ABC$. P is a point on (O,r) and P_1,P_2,P_3 are the projections of P onto BC, CA, AB. Circles $\mathcal{T}_1,\mathcal{T}_2$ and $\mathcal{T}_3$ touch BC,CA,AB through P_1,P_2,P_2 and (O,R) (internally), their centers lie on different sides of BC,CA,AB with respect to A,B,C. Prove that the sum of the lengths of the common external tangents of $\mathcal{T}_1,\mathcal{T}_2$ and $\mathcal{T}_3$ is a constant value.

Solution. Let δ_1 denote the tangent segment from A to $\mathcal{T}_1$. By Casey's theorem for $(A),(B),(C),\mathcal{T}_1$, all tangent to (O,R), we have $L \cdot BP_1 + L \cdot CP_1 = \delta_1 \cdot L \Longrightarrow \ \delta_1 = L$. Similarly, we have $\delta_2=\delta_3=L$. By Euler's theorem for the pedal triangle $\triangle P_1P_2P_3$ of P, we get:

$$[P_1P_2P_3] = \frac{p(P,(O))}{4R^2}[ABC] = \frac{R^2-r^2}{4R^2}[ABC] = \frac{3}{16}[ABC]$$

Therefore, we obtain

$$AP_2 \cdot AP_3 + BP_3 \cdot BP_1 + CP_1 \cdot CP_2 = \frac{2}{\sin 60^\circ}\left([ABC]-[P_1P_2P_3]\right) = \tfrac{13}{16}L^2. \ (\star)$$

By Casey's theorem for $(B),(C),\mathcal{T}_2,\mathcal{T}_3$, all tangent to (O,R), we get

$$\delta_2 \cdot \delta_3 = L^2 = BC \cdot \delta_{23} + CP_2 \cdot BP_3 = L \cdot \delta_{23} + (L-AP_1)(L-AP_2)$$

By cyclic exchange, we have the expressions:

$$L^2 = L \cdot \delta_{31} + (L-BP_3)(L-BP_1) \ , \ L^2 = L \cdot \delta_{12} + (L-CP_1)(L-CP_2)$$

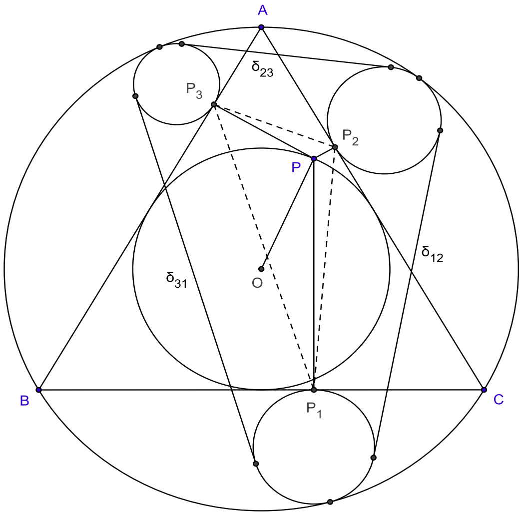

Figure 3: Application VII

Adding the three latter equations yields

$$3L^2 = L(\delta_{23} + \delta_{31} + \delta_{12}) + 3L^2 - 3L^2 + AP_3 \cdot AP_2 + BP_3 \cdot BP_1 + CP_1 \cdot CP_2$$

Hence, combining with $(\star)$ gives

$$\delta_{23} + \delta_{31} + \delta_{12} = 3L - \frac{13}{16}L = \frac{35}{16}L$$

3 Proposed Problems.

1) *Purser's theorem*: $\triangle ABC$ is a triangle with circumcircle (O) and ω is a circle in its plane. AX, BY, CZ are the tangent segments from A, B, C to ω. Show that ω is tangent to (O), if and only if

$$\pm AX \cdot BC \pm BY \cdot CA \pm CZ \cdot AB = 0$$

2) Circle ω touches the sides AB, AC of $\triangle ABC$ at P, Q and its circumcircle (O). Show that the midpoint of $\overline{PQ}$ is either the incenter of $\triangle ABC$ or the A-excenter of $\triangle ABC$, according to whether $(O), \omega$ are internally tangent or externally tangent.

3) $\triangle ABC$ is A-right with circumcircle (O). Circle Ω_B is tangent to the segments $\overline{OB}, \overline{OA}$ and the arc AB of (O). Circle Ω_C is tangent to the segments $\overline{OC}, \overline{OA}$ and the arc AC of (O). Ω_B, Ω_C touch $\overline{OA}$ at P, Q, respectively. Show that:

$$\frac{AB}{AC} = \frac{AP}{AQ}$$

4) *Gumma, 1874.* We are given a cirle (O, r) in the interior of a square $ABCD$ with side length L. Let (O_i, r_i) $i = 1, 2, 3, 4$ be the circles tangent to two sides of the square and (O, r) (externally). Find L as a fuction of r_1, r_2, r_3, r_4.

5) Two parallel lines τ_1, τ_2 touch a circle $\Gamma(R)$. Circle $k_1(r_1)$ touches Γ, τ_1 and a third circle $k_2(r_2)$ touches Γ, τ_2, k_1. We assume that all tangencies are external. Prove that $R = 2\sqrt{r_1 \cdot r_2}$.

6)*Victor Thébault. 1938.* $\triangle ABC$ has incircle (I, r) and circumcircle (O). D is a point on $\overline{AB}$. Circle $\Gamma_1(r_1)$ touches the segments $\overline{DA}, \overline{DC}$ and the arc CA of (O). Circle $\Gamma_2(r_2)$ touches the segments $\overline{DB}, \overline{DC}$ and the arc CB of (O). If $\angle ADC = \varphi$, show that:

$$r_1 \cdot \cos^2 \frac{\varphi}{2} + r_2 \cdot \sin^2 \frac{\varphi}{2} = r$$

References

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Stelling 3.2. *(De vlinderstelling) Laat M het midden zijn van een koorde PQ van een cirkel en AB en CD twee andere koorden door M. Noem X het snijpunt zijn van AD en PQ en Y van BC en PQ. Dan is M het midden van XY.*

Stelling 3.3. *(stelling van Pappos)*

Deze stelling luidt: Liggen A_1, B_1 en C_1 op een rechte d_1 en liggen A_2, B_2 en C_2 op een rechte d_2 , dan zijn de punten

A : snijpunt van B_1C_2 en B_2C_1, B : snijpunt van A_1C_2 en A_2C_1 en C : snijpunt van A_1B_2 en A_2B_1 collineair.

Stelling 3.4. *(stelling van Pascal)*

Neem zes willekeurige punten op een cirkel of andere kegelsnede, zeg $A, B, C, D, EenF$. Het snijpunt van delijnen AB en DE noemen we P, het snijpunt van BC en EF noemen we Q en het snijpunt van CD en FA noemen we R. Dan liggen P, Q en R op 1 lijn.

Stelling 3.5. *(gegeneraliseerde stelling van Pascal door Mobius) stel dat een veelhoek met $4n + 2$ zijden ingeschreven wordt in een kegelsnede, en paren van tegenoverstaande zijden worden verlengd totdat zij elkaar ontmoeten in $2n + 1$ punten, dan zal, als $2n$ van die punten op 1 lijn liggen, het laatste punt ook op die lijn liggen.*

Stelling 3.6. *(stelling van Brianchon)*

Neem een zeshoek $ABCDEF$ van zes raaklijnen aan een kegelsnede. Dan zijn de lijnenAD, BE en CF concurrent.

Stelling 3.7. *(stelling van Desargues)*

Twee driehoeken, ABC en XYZ, noemen we puntperspectief als AX, BY en CZ door 1 punt gaan en we noemen ze lijnperspectief als de snijpunten van AB en XY , BC en YZ, en CA en ZX op 1 lijn liggen. De stelling van Desargues zegt dat twee driehoeken lijnperspectief zijn dan en slechts als ze puntperspectief zijn.

Stelling 3.8. *(Poncelet) Als er een veelhoek tegelijk ingeschreven is in kegelsnede Γ_1 als kegelsnede Γ_2 omschrijft, bestaan er oneindig veel zo'n veelhoeken.*

Stelling 3.9. *(Taylorcirkel)*

Laat D, E, F de voetpunten zijn van A, B, C en zij D_1, D_2 de voetpunten van de loodlijnen uit D op AC, AB en analoog, dan gaat de Taylorcirkel door $D_1, D_2, E_1, E_2, F_1, F_2$.

Stelling 3.10. *(Morley's driehoek)*

De eerste snijpunten van de trisectrices vormen in iedere driehoek een gelijkzijdige driehoek.

Stelling 3.11. *(Steiner) de stelling van Steiner: In een $\triangle ABC$ geldt dat als $D, E \in [BC]$ en AD, AE isogonaal geconjugeerd zijn, geldt dat $\frac{|BD||BE|}{|DC||EC|} = \frac{|AB|^2}{|CA|^2}$*

resultaat van Steiner: de n-hoek met de grootste oppervlakte ingeschreven in een cirkel, is de regelmatige n-hoek

Stelling 3.12. *(De cirkel van Apollonius)*

Zij $[AB]$ *een lijnstuk en* k *een positief reel getal ongelijk aan* 1. *De meetkundige plaats van alle punten* P *waarvoor geldt* $\frac{|PA|}{|PB|} = k$ *is een cirkel met middelpunt op de rechte* AB.

Stelling 3.13. *(Gauss) Een rechte snijdt de zijden van een driehoek in punten* A', B', C' *, dan geldt dat de middens van* $[AA'], [BB'], [CC']$ *collineair zijn.*

Stelling 3.14. *(Brahmagupta)*

de formule: De oppervlakte in een vierhoek $ABCD : S = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2 \frac{\angle ABC + \angle BC}{2}}$
(vooral gekend als die cosinus 0 *is bij een koordenvierhoek, als generalisatie van de formule van Heroon)*

de stelling: In een koordenvierhoek $ABCD$ *waarvan de diagonalen loodrecht op elkaar staan in* S*, snijdt de loodlijn van* AB *door* S *de zijde* CD *in het midden (simpele angle-chasing)*

4 enkele speciale dingen

inversie

Stelling 4.1. *(inversie)*

Bij inversie wordt een punt O als centrum gekozen en ieder punt X wordt getransformeerd naar een punt Y zodat O, X, Y op de zelfde halfrechte liggen en $|OX||OY| = c$ waarbij c een reel getal is.

Indien f de inverterende functie is binnen deze meetkunde, geldt $f(X) = Y, f(Y) = X$ in dit voorbeeld, wat algemeen logisch $f(f(X)) = X$ heeft voor ieder voorwerp.

We zullen vanaf nu voor ieder punt A $f(A) = A'$ noteren om de eigenschappen op te sommen:

1. *een lijn door O wordt op zichzelf afgebeeld*
2. *een cirkel door O wordt geprojecteerd op een lijn die 0 niet bevat*
3. *een cirkel die niet door O gaat, wordt geprojecteerd op een andere cirkel die niet door O gaat.*
4. *hoeken worden behouden, maar er geldt wel dat $\angle OAB = \angle OB'A'$*
5. *lengtes van lijnstukken veranderen in volgende verhouding: $|A'B'| = \frac{|c||AB|}{|OA||OB|}$*

Met deze eigenschappen kunnen problemen vanuit een heel andere hoek worden opgelost en op een zeer ingenieuze manier opgelost worden.

Er is nog een andere transformatie om bepaalde gevallen simpeler te maken (opgelet met zo'n transformaties te combineren!!!)

Stelling 4.2. *(affiene meetkunde)*

Een affiene transformatie bestaat uit een afbeelding $(x, y) \to (ax + by + c, dx + ey + f)$.

Binnen de affiene meetkunde kunnen we met zo'n afbeelding 3 niet-collineaire punten vervangen door 3 andere niet-collineaire punten op een manier zoals je ze zelf kiest.

De affiene transformaties behouden

- *evenwijdigheid van lijnen*
- *collineairiteit van punten*
- *concurrentie van lijnen*

De hoeken als ook verhouding van lijnen zijn niet strikt noodzakelijk behouden.

Deze transformatie kan dus enkel helpen wanneer 1 van de andere 3 punten te bewijzen valt.