lesbrieffinale JEMC (1e editie)

olympia

november 2012

Inhoudsopgave

inleiding

Deze PDF bevat enkel de theorie die bij de junior EMC nuttig is om zo een kort overzicht (pagina 6 tot 18) te kunnen houden van hetgeen echt noodzakelijk is voor die competities.

Op Olympia kan men de **JBaMO** en **JEMC** bekijken als oefeningen, wat we als het belangrijkste beschouwen na dat volgende theorie verstaan is.

Veel succes en plezier met de mooie problem-solving die te ontdekken valt.

Merk op dat de info van de lesbrief over de VWO ook geldt, zoals grondig gevalonderscheid dat ook hier soms vragen kan oplossen zonder enige theorie.

werking

De bedoeling is dat volgende onderwerpen worden doorlopen door de theorie volledig door te nemen.
Op Olympia kan men de 2 competities (JBaMO en JEMC) bekijken om het echte probelm-solving in te oefenen met vragen van het juiste niveau gemengd.
Het is belangrijk dat men de voorbeelden en theorie goed snapt en de tijd durft te nemen om lang genoeg te proberen de oefeningen op te lossen voor de echte ervaring.
Wie oefeningen wil per onderwerp, kan de volledige bundel bekijken.
Indien een onderwerp onduidelijk is, kan men een link naar completer bestand vinden over dat onderwerp met meer voorbeelden
(soms ook al theorie die verder in het hoofdstuk stond of een competitievraag die ook in de vragenlijst stond, die dan natuurlijk niet zelf meer moet worden ingezonden als ze er nog niet opgelost was).
Het kan gebeuren dat zo'n bijlage in het Engels is en heel algemeen/moeilijk en delen bevat die minder interessant zijn.

De volgorde van de onderwerpen apart kan men kiezen en makkelijk terug vinden (bvb. via de inhoudstabel)

combinatoriek
algebra en analyse
getaltheorie
meetkunde

**Bij raad, opmerkingen of vragen ivm de bundel,
kan men dit via olympia@problem-solving.be melden.**

De nieuwste versie van deze bundel is dan te vinden op **met aanpassingen**

1 problem-solving inleiding

Problem-solving is iets meer dan gewoon uitrekenenen.

Merk trouwens op dat een vraag met identieke oplossing niet steeds even makkelijk is:

Bewijs dat 2^k een veelvoud geeft dat enkel bestaat uit enen en tweeen.
Men kan hier direct een inductie op het aantal factoren 2 uitvoeren.

Bewijs dat 2^k een veelvoud geeft dat geen nullen bevat.
Wanneer men dit wil bewijzen voor een getal als 2^{45} , zit er niet direct een logica achter omdat er heel veel mogelijkheden zijn.

Soms kan een extra voorwaarde de vraag dus zelfs vergemakkelijken en moet men zich niet laten afschrikken door lange vragen.

Het bewijzen van een vraag wordt meestal gedaan door logische argumenten op te sommen die leiden tot de oplossing.

Er zijn echter wat strategieen om de vragen op te lossen die we uitleggen en verschillende manieren van bewijs,
samen met wat basistheorie die frequent toegepast wordt.

We leggen kort uit hoe men een problem-solving-vraag kan oplossen a.h.v. een strategie van een internationaal gekend bestand dat uitgewerkt is met een eenvoudig IMO-voorbeeld.

Voorbeeld 1.1. *Bewijs dat de breuk* $\frac{21n+4}{14n+3}$ *voor geen enkel natuurlijk getal* n *vereenvoudigbaar is.*

stap 1: het begrijpen

Men bedoelt hier dat $\frac{21 \cdot n+4}{14 \cdot n+3}$ *een breuk is die niet vereenvoudigbaar is voor ieder natuurlijk getal dat we invullen.*

Voor $n = 1$ *hebben we de breuk* $\frac{25}{17}$ *dat inderdaad onvereenvoudigbaar is.*

We moeten het echter bewijzen voor alle natuurlijke getallen en kunnen dus niet alle gevallen afgaan.

stap 2: ideeen krijgen

Een breuk is onvereenvoudigbaar als de noemer en teller niet kunnen geleeld worden door een getal > 1*, wat betekent dat ze onderling priem moeten zijn.*

We zullen dus bewijzen dat de noemer en teller relatief priem zijn.

stap 3: ideeen uitwerken

TB: $ggd(21n + 4, 14n + 3) = 1$

als $r|21n + 4$ *dan is* $r|(21n + 4) \cdot m$ *waarbij* m *geheel is.*

Zo geldt ook als $r|21n + 4$ *dat* $r|(21n + 4) \cdot p$ *waarbij* p *geheel is.*

Het verschil van 2 veelvouden van r *is zelf ook een veelvoud van* r *en dus kunnen we* $r|(21n + 4)m - (14n + 3)p$ *uitwerken,*
daar $|(21n+4)m-(14n+3)p| \geq r$ *als het verschillend van 0 is, hebben we een voorwaarde voor* r.

$|(21n + 4)m - (14n + 3)p| = 1$ *bewijst ons gevraagde.*

We merken op dat $m = 2$ *en* $p = 3$ *werkt.*

stap 4: controle en oplossing uitschrijven

We controleren of onze ideeen voldoende zijn om het bewijs sluitend te maken:

bewijs

$ggd(21n + 4, 14n + 3)|3(14n + 3) - 2(21n + 4) = 1$, *waarbij we opmerken dat* $14n + 3$ *en* $21n + 4$ *beide verschillend van 0 zijn.*
Dit betekent dat $ggd(21n + 4, 14n + 3) = 1$

Omdat de teller en noemer relatief priem zijn, is $\frac{21n+4}{14n+3}$ *een onvereenvoudigbare breuk voor alle natuurlijke getallen* n.

Het echte bestand kan je vinden in het Engels in bijlage via problem-solvingstrategieen

2 combinatoriek + algemene problem-solving

2.1 basis

dubbeltellen

Men kan bepaalde eigenschappen combinatorisch bekijken om eigenschappen elegant te bewijzen.

* Het is belangrijk dat men dus de klassieke formules om te tellen kent:
*k elementen in volgorde plaatsen met keuze uit n elementen kan op $\frac{n!}{(n-k)!}$ manieren,
*indien elementen meerdere keren mogen voorkomen, hebben we n^k manieren om rijen te vormen van k elementen
*indien de volgorde niet belangrijk is, hebben we $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ manieren om k elementen te selecteren uit n waarden
*het aantal permutaties van een set $\{a_1, a_2, \cdots, a_s\}$ waarbij a_i k_i keer voorkomt en er in totaal n elementen zijn, is gelijk aan $\frac{n!}{k_1!k_2!\cdots k_s!}$

Deze linken op 2 manieren aan een zelfde probleem, kan leiden naar een contradictie of een ongelijkheid.

extremaalprincipe

Men bekijkt het kleinste of grootste element van een verzameling en door naar bepaalde eigenschappen te kijken of bewerkingen uit te voeren,
zien we dat er een groter/ kleinere waarde is, zodat ons extremum fout is, waardoor er ∞ veel elementen waarden zijn OF de vraag onmogelijk is.

We kunnen een oneindige afdaling doen bvb. om te zien dat er geen enkele waarde is die voldoet.

identiteiten

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$

$n^4 + 4x^4 = (n^2 + 2x^2 - 2xn)(n^2 + 2x^2 + 2xn)$ (identiteit van Sophie-Germaine)

$ap + bq$ met $ggd(p, q) = 1$ en $a, b \in \mathbb{N}$ kan alle waarden groter dan $pq - q - p$ aannemen, $pq - p - q$ is de grootste waarde die niet zo te schrijven is. (postzegelidenteit)

2.2 bedekkingen

Soms wordt in een vraag de mogelijkheid om iets te bedekken gevraagd.

Door een kleuring te gebruiken (enkele vakken in groepen verdelen) en eigenschappen zoals de pariteit te bekijken per object,
probeert men te bewijzen dat het al dan niet kan.

Voorbeeld 2.1. *Bewijs dat een $m*n$rechthoek enke volledig bedekt kan worden met $d*1$-blokken als $d|m$ en/ of $d|n$.*

Bewijs. Het is triviaal dat $d|mn$ moet gelden.

Bekijk het rechthoek als de unie van roosterpunten, zodat $(1,1),(1,n),(m,n),(m,1)$ de hoekpunten zijn.

Kleur (i,j) met een kleur $t \equiv i+j \pmod d$ zodat $t \in \{1,2,\cdot,d\}$ zit.

Het is duidelijk dat ieder $d*1$ blokje nu ieder kleur exact 1 keer bedekt.

Er moet dus gelden dat ieder kleur even vaak voorkomt, wat niet zo is:

Zij $m = kd+p$ en $n = dl+q$, dan bevat het $m*dl$ bord ieder kleur even vaak, alsook het $kd*q$-bord.

Het overige $p*q$ bord kan onmogelijk ieder kleur even vaak hebben.
Omdat $d|pq$ moeten $ggd(d,q)$ en $ggd(d,p) > 1$ zodat het volgende geldt:

Als $p+q \leq d$ komt het kleur d er niet voor of slechts 1 keer terwijl $pq > d$.
Als $p+q > d$ komt het kleur d er $q+p+1-d$ keer voor, terwijl kleur $d-1$ er $q+p+2-d$ keer voorkomt.
Contradictie en dus komt niet ieder kleur even vaak voor en is er geen bedekking mogelijk.

□

2.3 inductie

Bij inductie wordt een uitdrukking bewezen voor alle natuurlijke getallen vanaf k, dit door het te bewijzen voor k. (inductiebasis IB)
Vervolgens als het geldt voor n, bewijst men dat het ook geldt voor $n+1$.

Bij volledige inductie bewijst men bij de tweede stap dat de vraag geldt voor $n+1$ als het waar was $\forall i \in \{k, k+1 \cdots, n\}$

Voorbeeld 2.2. *(kleine stelling van Fermat) Er geldt dat* $n^p \equiv n \pmod{p}$ *als* p *priem is* $\forall n \in \mathbb{N}$.

Bewijs. Voor $n = 0$ en 1 is de vraag de triviaal. (IB)

Als de vraag geldt voor n, bewijzen we dat $(n+1)^p \equiv n+1 \pmod{p}$.

Lemma: Er geldt dat $p|\binom{p}{i}$ als $0 < i < p$ omdat $i!, (p-i)!$ geen factoren p bevatten.

$(n+1)^p = n^p + 1 + \sum_{i=1}^{p-1} \binom{p}{i} n^i \equiv n^p + 1 \equiv n+1 \pmod{p}$ door de inductiebasis en ons lemma. Met inductie geldt de stelling van Fermat nu voor alle getallen $n \in \mathbb{N}$.

□

2.4 invariantie en contradictie

Wanneer men wil bewijzen dat iets niet kan bij een combinatorische vraag, zijn er enkele manieren die vaak werken:

I Men zegt vanuit het ongerijmde dat de vraag wel kan opgelost worden en door de eigenschappen van de oplossing te bekijken,
bekomt men een contradictie waardoor er geen oplossing kon zijn (het ongerijmde was fout)

II Men bekijkt een eigenschap die invariant is in de vraag, waarbij die eigenschap bij de start en het einde verschillend is, waaruit volgt dat we het einde nooit kunnen bereiken.

III Men gebruikt een eigenschap die monotoon is bij iedere stap met een minimaal verschil, wanneer de eigenschap begrensd is, zijn er slechts een eindig aantal oplossingen.

opmerking Natuurlijk moet je opletten als je blijft proberen te bewijzen dat het niet werkt, dat er niet gewoon wel een oplossing was.

Voorbeeld 2.3. *We hebben de getallen van* 1 *tot* 2012^9 *in een pot gestoken.*
Iedere keer als we 2 *getallen* x *en* y *eruit halen, worden ze vervangen door* $(x-1006)(y-1006)+1006$ *en steken dit ene getal terug in de pot.*
Welk getal kunnen we vinden als er slechts 1 *getal meer in de pot zit?*

Bewijs. Merk op dat het getal 1006 en x vervangen wordt door 1006 en dit getal invariant blijft (in de pot terug wordt gestoken).

Dit getal blijft dus in de pot en zal het laatste getal 1006 zijn.

□

2.5 duivenhokprincipe (DVH-principe)

Zijn $n, k \in \mathbb{N}_0$.
Als men n duiven verdeelt over k duivenhokken, dan bestaat er een duivenhok dat minstens $\lfloor \frac{n-1}{k} \rfloor + 1$ duiven bevat.

Voorbeeld 2.4. *Binnen een cirkel met straal 16 liggen 650 gegeven punten.*
Definieer een ring als het vlakdeel dat begrepen is tussen twee concentrische cirkels met stralen 2 en 3 respectievelijk.
Bewijs dat men een ring kan plaatsen zodat minstens 10 van de 650 punten bedekt worden door deze ring.

Bewijs. Maak de cirkel met straal 16 nog iets groter tot een straal van 19.
Teken rond ieder van de 650 punten een ring.
De som van de oppervlakten van de ringen is $650 * 5\pi = 3250\pi$ en deze liggen allen in de cirkel met oppervlakte 361π.
Toevallig is $3250 = 9 * 361 + 1$ zodat er wegens 't DVH-principe een punt is dat in 10 ringen ligt.
Wanneer men nu een ring legt met centrum dat punt, waren er min. 10 centra van die 650 punten op een afstand tussen 2 en 3 zodat ze op onze geplaatste cirkel liggen.
Hiermee is het gevraagde bewezen.

□

2.6 winnende strategieen

Bij een vraag moeten we bewijzen dat iemand een winnende strategie heeft bij een spel,
dit kan door een kleuring of modulorekenen of andere elegante eigenschappen die worden uitgebuit.

stelling van Zermelo

Deze stelling zegt dat ieder spel tussen 2 personen waar toeval niet in meespeelt, geen gelijkspel mogelijk is en de spelers elk op hun beurt een zet doen:
1 van de 2 spelers geeft dan een winnende strategie.

Voorbeeld 2.5. *(QED-competitie)*

ALbert en Philip bestellen een zak met 2011 *frieten. Albert start met het eten van enkele frieten en eet om zijn beurt met Philip. Ze vorken* $1, 2, 5$ *frietjes op per keer. Degene die de laatste friet opeet, betaalt de rekening. Bewijs dat Philip zijn portemonnee kan laten zitten.*

Bewijs. Als Albert 1 frietje neemt, neemt Philip er 2.
Nam Albert er 2 of 5 neemt Philip er 1. Op die manier is er na Philip's beurt steeds een 3voud opgegeten, zodat hij niet de 2011^{de} friet at.

□

2.7 meetkunde binnen de combinatoriek

Het gebeurt vaak dat er interessante vraagjes over een meetkundige constructie plaats vindt op een grote olympiade.

Men kijkt naar specifieke eigenschappen die vaak logisch zijn en simpel te bewijzen zijn in een lemma.

Een logische stelling is de volgende

Stelling 2.6. *(tapijtenstelling)*

Wanneer enkele tapijten die samen de oppervlakte van de kamer hebben gelegd worden, is de oppervlakte die dubbel gelegd werd, gelijk aan de oppervlakte die niet bedekt werd.
Indien iets $m > 2$ keer belegd werd, moet je wel $m - 1$ keer de oppervlakte rekenen.

3 algebra

3.1 ongelijkheden

Stelling 3.1. *(extremeaaltechniek)*

Als een functie f convex of lineair is in alle variabelen $x_1, \cdots, x_n$, geldt dat het maximum optreedt wanneer alle variabelen gelijk zijn aan het minimum of maximum.

Stelling 3.2. *(AM-GM-HM) Voor $n \in \mathbb{N}$ $a_1, ..., a_n > 0$ geldt:*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}.$$

hints:

* Kwadraten zijn positief
* ontbindingen
* maximum/minimum beschouwen van de variabelen
* zorgen dat de gelijkheidsgevallen niet verdwenen zijn en in die gevallen nog gelijkheid blijft gelden
(men mag dus de vraag niet te veel vereenvoudigen dat de laatste stappen niet waar meer zijn)
* homogeniseren; zorgen dat de graad van iedere veelterm gelijk is
om zo passende voorwaarden te mogen stellen en de gewone stellingen te kunnen toepassen
* substituties: vervangen van de variabelen door een combinatie van nieuwe variabelen om de ongelijkheid te vereenvoudigen

Stelling 3.3. *(Cauchy-Schwarz [CS])*

Voor $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$ geldt:

$$(a_1^2 + a_2^2 + ... + a_n^2) \cdot (b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1 b_1 + ... + a_n b_n)^2.$$

Gelijkheid treedt op als en slechts als $\rho \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} \leq 1$.
(de waarde $\frac{a_i}{b_i}$ constant is voor alle i)

Dit kan uiteraard worden vervormd in andere vormen zoals:

(Cauchy-Schwarz in Engelvorm) Voor $a_1, ..., a_n \in \mathbb{R}$ en $b_1, ..., b_n > 0$ geldt:

$$\left(\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n}\right) \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + ... + b_n}.$$

(Holder-geval)

$$(a_1^3 + a_2^3 + ... + a_n^3) \cdot (b_1^3 + b_2^3 + ... + b_n^3) \cdot (c_1^3 + c_2^3 + ... + c_n^3) \geq (a_1 b_1 c_1 + ... + a_n b_n c_n)^3.$$

Stelling 3.4. *(Orde-ongelijkheid) Zij $a_1 \geq ... \geq a_n \geq 0$ en $b_1 \geq ... \geq b_n \geq 0$.*

Dan is

$$\prod (a_i + b_{n+1-i}) \geq \prod (a_i + b_j) \geq \prod (a_i + b_i)$$

en

$$\sum a_i b_i \geq \sum a_i b_j \geq \sum a_i b_{n+1-i}$$

4 getaltheorie

4.1 basis

De basis bij getaltheorie bestaat uit o.a. de eenduidige priemontbinding, kgv, ggd, aantal delers en dergelijke kennen en toepassen.

De stelling van Bezout-Bachet zegt dat $\forall a, b \in \mathbb{N}$ er $n, m \in \mathbb{N}$ bestaan zodat $ggd(a, b) = an + bm$.

kleine stelling van Fermat $p|a^p - a$

De stelling van Euler zegt dat $n^{\phi m} \equiv 1 \pmod{m}$ als $ggd(n, m) = 1$.
Als $m = \prod p_i^{k_i}$ is $\phi(m) = \prod (p_i - 1)p_i^{k_i - 1}$.

Herbij is $a \equiv b \pmod{m}$ aesa $m|a - b$

chinese reststelling (CRS) Als m_1 tot m_k gehele getallen die paarsgewijs relatief priem zijn en a_1 tot a_k zijn gehele getallen.
Dan bestaat er 1 oplossing x modulo $\prod m_i$ zodat $x \equiv a_i \pmod{m_i}$ $\forall i \in \{1, 2 \cdots, k\}$.

stelling van Wilson Voor ieder priemgetal geldt $(p - 1)! \equiv -1 \pmod{p}$.

Indien er nog vragen waren over de basis, is volgende bestand altijd handig om te helpen in dit getaltheorie-hoofdstuk (voor dit deel vooral de eerste 2 pagina's : getallenleer)

Nog een extra lemma is LiftingTheExponent

5 meetkunde

5.1 basis (herhaling)

Een driehoek $\triangle ABC$ is gelijkbenig met $|AB| = |AC|$ als en slechts als de twee hoeken ("basishoeken") B en C gelijk zijn.

In gelijkvormige driehoeken $\triangle ABC$ en$\triangle XYZ$ zijn de overeenkomstige hoeken gelijk en de overeenkomstige zijden hebben een constante verhouding ("de zijden zijn evenredig").
I.e.$A = X, B = Y, C = Z, \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$. Notatie $ABC \sim XYZ$

Twee driehoeken $\triangle ABC$ en $\triangle XYZ$ zijn congruent (gelijkvormig en de overeenkomstige zijden zijn even lang) als een van de volgende voldaan is ("congruentiekenmerken")
Alle overeenkomstige zijden even lang zijn ("ZZZ")
Twee overeenkomstige zijden even lang zijn, en de ingesloten hoeken gelijk zijn ("ZHZ")
Twee overeenkomstige hoeken gelijk zijn, en n overeenkomstig paar zijden even lang is ("HZHën "ZHH")

Voor gelijkvomigheid is het voldoende dat de verhouding van de zijden gelijk is (ZHH wordt dan HH)

Stelling van Pythagoras: in een rechthoekige driehoek met rechte hoek A geldt $|AB|^2 + |AC|^2 = |BC|^2$.

Als twee driehoeken een gemeenschappelijke top hebben, en een basis met dezelfde drager (de drager van een lijnstuk AB is de rechte AB), dan verhouden hun oppervlakten zich als de lengten van hun basissen.
Dus: voor driehoeken ABC en ADE, met B, C, D, E colineair (op dezelfde rechte) geldt $[ABC]/[ADE] = BC/DE$.
Dit volgt onmiddellijk uit öppervlakte driehoek = basis * hoogte / 2". Hierbij staat $[ABC]$ voor de oppervlakte van de driehoek ABC.

Nog wat naamkennis herhalen:

* De zwaartelijnen van een driehoek (uit een hoekpunt naar het midden van de overstaande zijde) snijden elkaar in 1 punt, het zwaartepunt Z van de driehoek, in de theorie staat er G
* De hoogtelijnen van een driehoek (uit een hoekpunt loodrecht op de overstaande zijde) snijden elkaar in 1 punt, het hoogtepunt H van de driehoek.
* De middelloodlijnen van een driehoek (de middelloodlijnen van de zijden) snijden elkaar in 1 punt, het omcentrum O van de driehoek.
* De bissectrices van een driehoek (de bissectrices van de hoeken) snijden elkaar in 1 punt, het incentrum I van de driehoek.

eig. Vliegers:

*2 paar aangrenzende zijden zijn even lang
*De diagonalen van een vlieger staan loodrecht op elkaar.

basiseigenschappen over koordenvierhoeken:

*de overstaande hoeken zijn supplementair

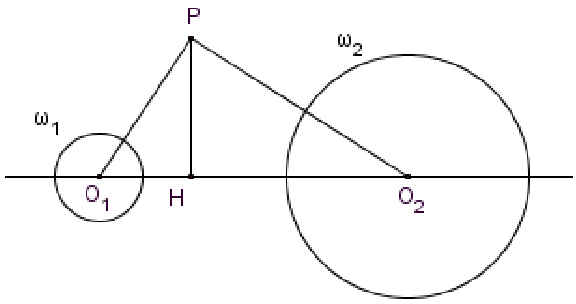
*omtrekshoeken op een gelijke boog zijn gelijk, met een waarde die de helft is van de middelpuntshoek op de boog.

5.2 Macht van een punt

Wanneer we een punt P en een cirkel ω met middelpunt O en straal R beschouwen, en we tekenen een willekeurige rechte door P die ω snijdt in A en B, dan merken we op dat de grootte van $|PA| \,.\, |PB|$ onafhankelijk is van de gekozen rechte.
We definiëren P_ω^P :de *macht van* P *t.o.v.* ω in het algemeen door $|OP|^2 - R^2$, maar het is in specifieke gevallen te berekenen met

- $-|PA| \,.\, |PB|$ als P binnen de cirkel ligt.
- 0 als P op de cirkel ligt.
- $|PA| \,.\, |PB|$ als P buiten de cirkel ligt.

Machtlijn



Figuur 1: De machtlijn

Wanneer er twee cirkels in het spel zijn zouden we ons kunnen afvragen wat de *meetkundige plaats* is van alle punten P die t.o.v. beide cirkels dezelfde macht hebben. Beschouw daartoe twee cirkels ω_1, ω_2 met straal r_1, r_2 en middelpunt O_1, O_2. Zij P een punt dat gelijke macht ten opzichte van beide cirkels heeft, en noem H de projectie van P op O_1O_2. De meetkundige plaats die we zochten is een rechte loodrecht op O_1O_2. Deze rechte wordt ook wel de *machtlijn* van beide cirkels genoemd.

Merk op dat we eenvoudig de machtlijn van twee snijdende cirkels kunnen terugvinden als de rechte door beide snijpunten (of de gemeenschappelijk raaklijn indien de cirkels raken in een punt). Ga na waarom dat zo is.

Machtpunt

Wanneer we drie cirkels beschouwen, dan kunnen we voor elk paar cirkels de machtlijn gaan beschouwen. Bewijs nu zelf de volgende stelling:

Gegeven zijn drie cirkels ω_1, ω_2 *en* ω_3*. De drie machtlijnen die we krijgen door telkens twee verschillende cirkels uit de gegeven drie cirkels te beschouwen zijn concurrent.*

Het punt van concurrentie van deze 3 machtlijnen wordt vaak het *machtpunt* van de drie cirkels genoemd.

5.3 nuttige dingen en enkele eigenschappen

- isometrieen zoals homotheties, verschuivingen, verdraaiingen
- gelijkvormigheid en congruentie
- de eigenschappen van koordenvierhoeken en omtrekshoeken
- vectoren.
- angle-chasing: hoekjagen, vaak start men met hoekjagen om hieruit de conclusies i.v.m. cirkelbogen te kunnen trekken
- constructie van interessante punten die helpen het probleem op te lossen.
  Wanneer men iets moet bewijzen dat niet direct meetkundig te interpreteren is, kan een constructie van een nieuw punt vaak helpen, bvb. als de som van 2 lengtes gelijk moet zijn aan 1 lengte.
  Bij collineairiteit, kan het helpen, wanneer men een nieuw punt P beschouwt, als P, A, B en P, A, C collineair zijn, zijn A, B, C ook collineair.
- Als het niet duidelijk is, hoe iets te bewijzen valt, moet men het goede idee/ oplossing soms zien van het meetkundig probleem en dan bewijzen dat die oplossing idd voldoet.

- (De rechte van Simson)

  Drie punten zijn collineair als en slechts als de driehoek gevormd door deze drie punten een oppervlakte heeft die nul is. Ga nu zelf met behulp van eigenschap 3 uit de vorige paragraaf de volgende stelling na:

  De projecties van een punt P op de zijden van ABC zijn dan en slechts dan collineair als P op de omgeschreven cirkel van ABC ligt.

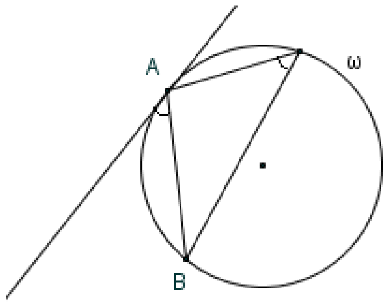
  De rechte die de drie projecties van het punt P bevat noemt men de *rechte van Simson* van punt P t.o.v. ABC.

  Deze stelling zal je soms van pas komen wanneer je een probleem te lijf gaat. Tracht ook als oefening eens een rechtstreeks bewijs te vinden, dus zonder de uitdrukking voor de oppervlakte van een voetpuntsdriehoek te gebruiken.

- **de Steinerlijn** is de lijn l gevormd door een punt P op de omgeschreven cirkel te spiegelen over AB, BC, AC en gaat door H.
  Het is de homothetie met center in P van de Simsonlijn met een factor 2.
  Het punt P wordt het antiSteinerpunt van l tov $\triangle ABC$ genoemd

Wie het toch nodig vindt, kan de uitgebreide bijlage basis meetkunde eerst eens doornemen.

In deze paragraaf bekijken we enkele lemma's van dichterbij die ongewoon vaak hun intrede deden in problem-solving-problemen de voorbije jaren.



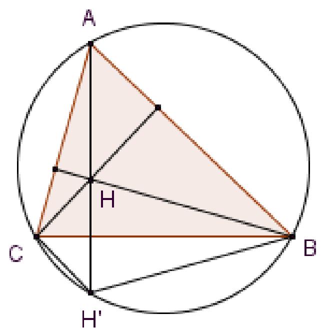
Figuur 2: De raakomtrekshoek

Lemma 1 (Raakomtrekshoek) *Beschouw een cirkel ω die de punten A en B bevat. De raaklijn aan ω in A sluit een hoek in met AB die in grootte gelijk is aan een van beide omtrekshoeken op AB in ω.*

(Bewijs als oefening)

Lemma 2 *De reflecties van het hoogtepunt H van ABC ten opzichte van de zijden liggen op de omgeschreven cirkel van ABC.*

Het bewijs van dit lemma is eenvoudig en kan als oefening dienen voor de lezer.

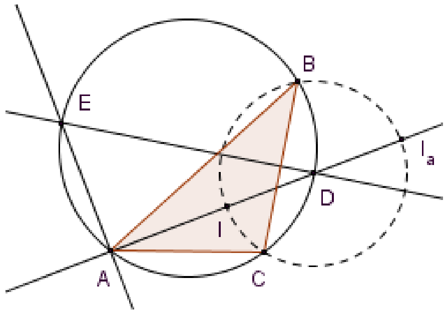


Figuur 3: Lemma 2

Lemma 3 *In driehoek ABC noemen we I het middelpunt van de ingeschreven cirkel, en I_a het middelpunt van de aangeschreven cirkel tegenover A.*

- *De binnen(resp. buiten)bissectrice van A snijdt de middelloodlijn van BC in het punt D (resp. het punt E) op de omgeschreven cirkel.*
- *De cirkel met diameter II_a bevat B en C en heeft D als middelpunt.*
- *De cirkel door B, C, I_c, I_b geeft E als middelpunt*

Dit lemma is allicht het belangrijkste uit deze hele paragraaf, en een van de vaakst terugkerende lemmata(zelfs bij IMO-problemen).
Het bewijs is een goede oefening angle-chasing, maar het is ook een gevolg van het feit dat ABC de negenpuntscirkel is van $I_aI_bI_c$.



Figuur 4: Lemma 3

Lemma 4 *In driehoek ABC zijn O en H isogonaal verwant*

Bewijs. met angle-chasing: $\angle BAH = \angle CAO = 90 - \alpha$ en analoog □

Voor zij die meer lemma's ivm meetkunde willen kennen :

meetkundelemma's

of projectieve meetkunde met meer synthetische stellingen willen hebben: projectieve meetkunde

6 bijlages en bronnen

Hier staan PDF's die uitgebreid handelen over een specifiek deel.
Er staan nog enkele nieuwe dingen tussen ter volledigheid.
Naar de interessante stukken wordt op het juiste moment verwezen doorheen de echte bundel.

6.1 problem-solvingstrategieen

Polya's Problem Solving Techniques

In 1945 George Polya published the book *How To Solve It* which quickly became his most prized publication. It sold over one million copies and has been translated into 17 languages. In this book he identifies four basic principles of problem solving.

Polya's First Principle: Understand the problem

This seems so obvious that it is often not even mentioned, yet studens are often stymied in their efforts to solve problems simply because they don't understand it fully, or even in part. Polya taught teachers to ask students questions such as:

- Do you understand all the words used in stating the problem?
- What are you asked to find or show?
- Can you restate the problem in your own words?
- Can you think of a picture or diagram that might help you understand the problem?
- Is there enough information to enable you to find a solution?

Polya's Second Principle: Devise a plan

Polya mentions that there are many reasonable ways to solve problems. The skill at choosing an appropriate strategy is best learned by solving many problems. You will find choosing a strategy increasingly easy. A partial list of strategies is included:

- Guess and check
- Make an orderly list
- Eliminate possibilities
- Use symmetry
- Consider special cases
- Use direct reasoning
- Solve an equation
- Look for a pattern
- Draw a picture
- Solve a simpler problem
- Use a model
- Work backwards
- Use a formula
- Be ingenious

Polya's Third Principle: Carry out the plan

This step is usually easier than devising the plan. In general, all you need is care and patience, given that you have the necessary skills. Persist with the plan that you have chosen. If it continues not to work discard it and choose another. Don't be misled, this is how mathematics is done, even by professionals.

Polya's Fourth Principle: Look back

Polya mentions that much can be gained by taking the time to reflect and look back at what you have done, what worked, and what didn't. Doing this will enable you to predict what strategy to use to solve future problems.

So starting on the next page, here is a summary, in the master's own words, on strategies for attacking problems in mathematics class. This is taken from the book, *How To Solve It*, by George Polya, 2nd ed., Princeton University Press, 1957, ISBN 0-691-08097-6.

1. UNDERSTAND THE PROBLEM

- **First.** You have to *understand* the problem.
- What is the unknown? What are the data? What is the condition?
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
- Draw a figure. Introduce suitable notation.
- Separate the various parts of the condition. Can you write them down?

2. DEVISING A PLAN

- **Second.** Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.
- Have you seen it before? Or have you seen the same problem in a slightly different form?
- *Do you know a related problem?* Do you know a theorem that could be useful?
- *Look at the unknown!* Try to think of a familiar problem having the same or a similar unknown.
- *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
- Could you restate the problem? Could you restate it still differently? Go back to definitions.
- If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3. CARRYING OUT THE PLAN

- **Third.** *Carry out* your plan.
- Carrying out your plan of the solution, *check each step.* Can you see clearly that the step is correct? Can you prove that it is correct?

4. LOOKING BACK

- **Fourth.** *Examine* the solution obtained.
- Can you *check the result*? Can you check the argument?
- Can you derive the solution differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

terug naar echt bestand

Getallenleer
IMO-stage Beersel 2011

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4 april 2011

1 Deelbaarheid en priemgetallen

We beginnen met een herhaling van de belangrijkste definities en eigenschappen. We laten de bewijzen daarbij achterwege, maar het is uiteraard een goed idee om te proberen om die bewijzen zelf te geven...

Definities

(1) Als $m, n \in \mathbb{Z}$, dan zeggen we dat m een deler is van n (notatie: $m \mid n$) als er een $a \in \mathbb{Z}$ bestaat zodat $n = am$.

(2) Een natuurlijk getal p is priem als p precies twee *positieve* delers heeft.

(3) Gegeven $a \in \mathbb{Z}$ en $b \in \mathbb{N}_0$, dan bestaan er twee (unieke) gehele getallen q en r zodanig dat $a = qb + r$ met $0 \leq r < b$. We noemen q en r respectievelijk het quotiënt en de rest bij deling van a door b.

(4) Zijn $m, n \in \mathbb{Z}$. De grootste gemene deler van m en n is het (unieke) natuurlijk getal d dat voldoet aan $d \mid m$, $d \mid n$ en de volgende voorwaarde: als e een geheel getal is met $e \mid m$ en $e \mid n$, dan is $e \mid d$. Het kleinste gemene veelvoud van m en n is het (unieke) natuurlijk getal a dat voldoet aan $m \mid a$, $n \mid a$ en de volgende voorwaarde: als b een geheel getal is met $m \mid b$ en $n \mid b$, dan is $a \mid b$. We noteren $d = \text{ggd}(m, n)$ en $a = \text{kgv}(m, n)$. Op analoge wijze kunnen we de grootste gemene deler en het kleinste gemene veelvoud van meer dan twee gehele getallen definiëren.

(5) Twee gehele getallen zijn onderling ondeelbaar als hun grootste gemene deler gelijk is aan 1.

Eigenschappen

(1) Als m en n gehele getallen zijn met $m \mid n$, dan is $|m| \leq |n|$.

(2) Elk natuurlijk getal kan op unieke wijze worden geschreven als een product van priemgetallen: gegeven een natuurlijk getal n, dan bestaan er priemgetallen $p_1, p_2, \cdots, p_r$ en natuurlijke getallen $a_1, a_2, \cdots, a_r \geq 1$ zodat $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, en deze schrijfwijze is uniek op permutatie van de factoren na.

(3) (Euclides) Er bestaat oneindig veel priemgetallen. Sterker nog (postulaat van Bertrand): voor elke $n \in \mathbb{N}_0$ bestaat er een priemgetal p met $n \leq p \leq 2n$. Een ander nuttig resultaat (stelling van Dirichlet): als a en b onderling ondeelbaar zijn, dan bestaan er oneindig veel priemgetallen van de vorm $an + b$.

(4) Als m en n natuurlijke getallen zijn, dan is $\text{ggd}(m, n) \cdot \text{kgv}(m, n) = mn$. De grootste gemene deler van twee natuurlijke getallen kan worden berekend met het algoritme van Euclides. Als $m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ en $n = p_1^{b_1} p_2^{b_2} \cdots p_r^{a_r}$ met $p_1, p_2, \cdots, p_r$ verschillende priemgetallen en $a_1, a_2, \cdots, a_r \geq 0$, dan is

$$\text{ggd}(m, n) = p_1^{\min(a_1, b_1)} p_2^{\min(a_2, b_2)} \cdots p_r^{\min(a_r, b_r)}, \quad \text{kgv}(m, n) = p_1^{\max(a_1, b_1)} p_2^{\max(a_2, b_2)} \cdots p_r^{\max(a_r, b_r)}.$$

(5) Als $a, b, c \in \mathbb{Z}$, $a \mid bc$ en $\text{ggd}(a, b) = 1$, dan $a \mid c$. Dus als $m, n \in \mathbb{Z}$, als p priem is en $p \mid mn$, dan is $p \mid m$ of $p \mid n$.

(6) Als $a, b, c \in \mathbb{Z}$ zodat $a \mid c$, $b \mid c$ en $\text{ggd}(a, b) = 1$, dan is $ab \mid c$.

(7) (Bézout) Zijn $a_1, a_2, \cdots, a_n \in \mathbb{Z}$. Dan bestaan er $m_1, m_2, \cdots, m_n \in \mathbb{Z}$ met $a_1m_1 + \cdots + a_nm_n = \text{ggd}(a_1, \cdots, a_n)$.

(8) Als $d, m, n \in \mathbb{Z}$ met $d \mid m$ en $d \mid n$, dan geldt ook dat $d \mid am + bn$ voor alle $a, b \in \mathbb{Z}$. Met andere woorden: een deler van twee gehele getallen is ook een deler van elke lineaire combinatie van deze twee getallen. Deze eigenschap kan makkelijk worden veralgemeend naar een gemeenschappelijke deler van drie of meer gehele getallen.

(9) Zij n een geheel getal met $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ als unieke ontbinding in priemfactoren. Zij $\tau(n)$ het aantal positieve delers van n en zij $\sigma(n)$ de som van de positieve delers van n. Dan hebben we de volgende gelijkheden:

$$\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1), \quad \sigma(n) = \left(\frac{p_1^{a_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{a_2+1} - 1}{p_2 - 1}\right) \cdots \left(\frac{p_r^{a_r+1} - 1}{p_r - 1}\right).$$

(10) Zij n een natuurlijk getal en p een priemgetal. De exponent van p in de priemfactorenontbinding van $n!$ is gelijk aan

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

met $\lfloor x \rfloor$ het grootste geheel getal kleiner dan of gelijk aan x.

Voorbeelden

Voorbeeld 1. Bepaal alle natuurlijke getallen n zodat $7^n \mid 9^n - 1$.

Oplossing. Merk op dat $9^n - 1 = (3^n - 1)(3^n + 1)$. Nu is $\text{ggd}(3^n - 1, 3^n + 1) = 2$, en daaruit volgt dat $7^n \mid 3^n - 1$ of $7^n \mid 3^n + 1$ (waarom?). Maar $7^n > 3^n + 1 > 3^n - 1$ voor $n \geq 1$, dus de enige oplossing is $n = 0$.

Voorbeeld 2. Zij $a, m, n \in \mathbb{N}$ met $a \geq 2$. Dan is $\text{ggd}(a^m - 1, a^n - 1) = a^{\text{ggd}(m,n)} - 1$.

Bewijs. Zij $d = \text{ggd}(m, n)$. Het is duidelijk dat $a^d - 1 \mid a^m - 1$ en $a^d - 1 \mid a^n - 1$, dus $a^d - 1 \mid \text{ggd}(a^m - 1, a^n - 1)$. Kies nu gehele getallen p en q met $pm + qn = d$. Veronderstel - zonder verlies van de algemeenheid - dat $p > 0$ en $q < 0$. Zij s een gemeenschappelijke deler van $a^m - 1$ en $a^n - 1$. We moeten nagaan dat $s \mid a^d - 1$. Maar $s \mid a^{pm} - 1$ en $s \mid a^{-qn} - 1$, dus $s \mid (a^{pm} - 1) - (a^{-qn} - 1)$, of nog, $s \mid a^{-qn}(a^{pm+qn} - 1)$. Maar $\text{ggd}(s, a) = 1$ (waarom?), dus $s \mid a^d - 1$. □

Voorbeeld 3. (IMO 1994) Bepaal alle koppels (m, n) van natuurlijke getallen m en n zodat $mn - 1 \mid m^3 + 1$.

Oplossing. Merk op dat $mn - 1 \mid n(m^3 + 1) - m^2(mn - 1)$, dus $mn - 1 \mid m^2 + n$. Stel $m^2 + n = a(mn - 1)$. Dan geldt $m^2 - amn + n + a = 0$. Bekijk de vierkantsvergelijking $x^2 - anx + a + n = 0$. Dan is m zeker een oplossing van de vergelijking - zij p de andere oplossing (het is mogelijk dat $p = m$). Dan geldt er $m + p = an$ en $mp = a + n$. Als $a, m, n, p \geq 2$, dan geldt $mp \geq m + p = an \geq a + n = mp$. Bijgevolg moet er gelijkheid optreden, dus $m = n = a = p = 2$.

Veronderstel dus dat één van de getallen a, m, n, p gelijk is aan 1. Stel eerst $a = 1$. Dan is $mp = n + 1$ en $m + p = n$, dus $mp = m + p + 1$, of nog, $(m - 1)(p - 1) = 2$. Bijgevolg is $(m, p) = (2, 3)$ of $(m, p) = (3, 2)$, en $n = 5$. Wegens symmetrie geeft de veronderstelling $n = 1$ dezelfde oplossingen voor (m, p). Veronderstel nu dat $p = 1$. Dan is analoog $(a, n) = (2, 3)$ of $(a, n) = (3, 2)$, en $m = 5$. Wegens symmetrie geeft de veronderstelling $m = 1$ dezelfde oplossingen voor (a, n).

Samenvattend: de mogelijke oplossingen zijn $(m, n) \in \{(2, 2), (2, 5), (3, 5), (5, 2), (5, 3), (1, 2), (1, 3), (2, 1), (3, 1)\}$, en een eenvoudige controle leert ons dat elk van deze koppels inderdaad een oplossing is. □

Voorbeeld 4. (IMO 2002) Zij $n \geq 2$ een natuurlijk getal met positieve delers $1 = d_1 < d_2 < \cdots < d_k = n$. Toon aan dat $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k < n^2$. Wanneer geldt $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k \mid n^2$?

Oplossing. Merk op dat $d_i d_{k+i-1} = n$. We moeten dus bewijzen dat

$$\frac{1}{d_1d_2} + \frac{1}{d_2d_3} + \cdots + \frac{1}{d_{k-1}d_k} < 1.$$

Maar $d_i \geq i$, dus

$$\frac{1}{d_1d_2}+\frac{1}{d_2d_3}+\cdots+\frac{1}{d_{k-1}d_k} \leq \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k-1)\cdot k} = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{k-1}-\frac{1}{k}\right) = 1-\frac{1}{k} < 1.$$

Nu is $d_2 = p$ priem (waarom?). Dan is $d_{k-1} = n/p$, dus $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k \geq d_{k-1}d_k = n^2/p$. Maar de grootste echte deler van n^2 is n^2/p. Dus $d_1d_2 + d_2d_3 + \cdots d_{k-1}d_k$ deelt n^2 a.s.a. $k = 2$, m.a.w. als $n = p$ een priemgetal is.

Opgaven

(1) Zijn x en y gehele getallen. Bewijs dat $17 \mid 2x + 3y$ als en slechts als $17 \mid 9x + 5y$.

(2) Bewijs dat het product van n opeenvolgende natuurlijke getallen steeds deelbaar is door $n!$.

(3) Zij $n \geq 5$ een natuurlijk getal. Bewijs: n is niet priem $\iff n \mid (n-1)!$.

(4) Zijn $a, b, m, n \in \mathbb{N}$ zodat $a^m - 1$ en $b^n + 1$ priem zijn. Geef zoveel mogelijk informatie over a, b, m en n.

(5) Zij $n \in \mathbb{N}$ zodat $24 \mid n + 1$. Bewijs dat de som van de positieve delers van n ook deelbaar is door 24.

(6) (IMO 1972) Bewijs dat de volgende uitdrukking een natuurlijk getal is voor alle $m, n \in \mathbb{N}_0$:

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}.$$

(7) Bewijs dat de volgende uitdrukking een natuurlijk getal is voor alle $m, n \in \mathbb{N}_0$:

$$\frac{\text{ggd}(m,n)}{n}\binom{n}{m}.$$

(8) (IMO 1992) Bepaal alle natuurlijke getallen $1 < a < b < c$ zodat $(a-1)(b-1)(c-1) \mid abc - 1$.

(9) (IMO 2009) Zij n een natuurlijk getal. Zijn $a_1, a_2, \cdots, a_k$ (met $k \geq 2$) verschillende elementen van de verzameling $\{1, 2, \cdots, n\}$ zodanig dat $n \mid a_i(a_{i+1} - 1)$ voor $i = 1, 2, \cdots, k-1$. Bewijs dat n geen deler is van $a_k(a_1 - 1)$.

(10) (IMO 1998) Bepaal alle natuurlijke getallen a en b zodat $ab^2 + b + 7 \mid a^2b + a + b$.

2 Modulo-rekenen

Gegeven gehele getallen a, b en m met $m \geq 2$, dan zeggen we dat a en b congruent zijn modulo m - of nog, $a \equiv b \pmod{m}$ - indien $m \mid a - b$, of nog, indien a en b dezelfde rest geven bij deling door m. Op deze manier krijgen we een equivalentierelatie (een transitieve, symmetrische en reflexieve relatie) op de verzameling $\mathbb{Z}$ van gehele getallen die ons toelaat om met "restklassen modulo m" te rekenen in plaats van met alle gehele getallen. Het grote voordeel van deze operatie is natuurlijk dat er slechts eindig veel restklassen modulo m zijn. . . De restklassen modulo m vormen een algebraïsche structuur die we de *ring* $\mathbb{Z}/m\mathbb{Z}$ noemen. Optelling en vermenigvuldiging zijn in deze ring gedefinieerd op de evidente manier: als $a \equiv b \pmod{m}$ en $c \equiv d \pmod{m}$, dan geldt $a + c \equiv b + d \pmod{m}$ en $ac \equiv bd \pmod{m}$. (Ga dat na!)

Voor een gegeven $a \in \mathbb{Z}$ en $m \geq 2$ zeggen we dat x een inverse is voor a modulo m als $ax \equiv 1 \pmod{m}$.

Bewering. Er bestaat een inverse voor a modulo m als en slechts als $\text{ggd}(a, m) = 1$.

Bewijs. Stel dat x een inverse is voor a. Dan is $ax \equiv 1 \pmod{m}$, m.a.w er bestaat een $p \in \mathbb{Z}$ met $ax = 1 + pm$. Dus $ax - pm = 1$. Maar $\text{ggd}(a, m)$ deelt $ax - pm$, dus $\text{ggd}(a, m) = 1$. Omgekeerd, als $\text{ggd}(a, m) = 1$, dan bestaat er volgens de stelling van Bézout een geheel getal p zodat $ax - pm = 1$, dus $ax \equiv 1 \pmod{m}$. □

Het aantal restklassen modulo m die een inverse hebben kan dus worden geïdentificeerd met de verzameling van natuurlijke getallen a met $0 < a < m$ en $\text{ggd}(a, m) = 1$. We noteren $\varphi(m)$ voor het aantal natuurlijke getallen a met die eigenschappen (φ is de *Euler-functie*). Indien $m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ de unieke ontbinding van m in priemfactoren is, dan geldt er

$$\varphi(m) = (p_1 - 1)p_1^{a_1-1}(p_2 - 1)p_2^{a_2-1} \cdots (p_r - 1)p_r^{a_r-1}.$$

(Oefening: probeer deze gelijkheid zelf af te leiden!)

Eigenschappen

(1) Zij $m \geq 2$ een natuurlijk getal en zij $a \in \mathbb{Z}$ met $\text{ggd}(a, m) = 1$. Er bestaat een kleinste natuurlijk getal $q \neq 0$ met de eigenschap dat $a^q \equiv 1 \pmod{m}$. Dit getal q heet de *orde* van a modulo m en is een deler van $\varphi(m)$. Als r een willekeurig natuurlijk getal is zodanig dat $a^r \equiv 1 \pmod{m}$, dan geldt $q \mid r$.

(2) (Fermat) Als p priem is en $a \in \mathbb{N}$ is niet deelbaar door p, dan is $a^{p-1} \equiv 1 \pmod{p}$. (Dit is een speciaal geval van (1)!)

(3) (Wilson) Als p priem is, dan is $(p - 1)! \equiv -1 \pmod{p}$.

(4) (Primitieve wortels) Zij $m \geq 2$. Een natuurlijk getal a waarvan de orde modulo m gelijk is $\varphi(m)$ noemen we een *primitieve wortel* modulo m. Een primitieve wortel bestaat als en slechts als $m = 2$, $m = 4$, $m = p^k$ of $m = 2p^k$ met p priem. Als a een primitieve wortel is modulo p en $a^{p-1} \not\equiv 1 \pmod{p^2}$, dan is a ook een primitieve wortel modulo p^2; indien $a^{p-1} \equiv 1 \pmod{p^2}$, dan is $a + p$ een primitieve wortel modulo p^2. Als a een primitieve wortel is modulo p^k met $k \geq 2$, dan is a ook een primitieve wortel modulo p^ℓ voor alle $\ell \geq k$.

(5) (Chinese reststelling) Zijn $m_1, m_2, \cdots, m_k$ gehele getallen die paarsgewijs onderling ondeelbaar zijn, m.a.w. zodat $\text{ggd}(m_i, m_j) = 1$ als $i \neq j$. Zijn $a_1, a_2, \cdots, a_k \in \mathbb{Z}$ willekeurig. Dan bestaat er een natuurlijk getal x zodat $x \equiv a_1 \pmod{m_1}$, $x \equiv a_2 \pmod{m_2}$, $\cdots$, $x \equiv a_k \pmod{m_k}$. Dat getal x is bovendien uniek modulo $m_1 m_2 \cdots m_k$.

(6) (Kwadraten) Volkomen kwadraten zijn congruent met 0 of 1 modulo 4, congruent met 0 of 1 modulo 3, congruent met 0, 1 of 4 modulo 8, ... In het algemeen geldt: als p een oneven priemgetal is, dan bestaan er precies $\frac{1}{2}(p+1)$ restklassen modulo p (inclusief 0) die een volkomen kwadraat zijn modulo p. Als p een willekeurig priemgetal is, dan is -1 een kwadraat modulo p als en slechts als $p = 2$ of $p \equiv 1 \pmod{4}$, en 2 is een kwadraat modulo p als en slechts als $p = 2$ of $p \equiv \pm 1 \pmod{8}$. Dus elke priemdeler van een natuurlijk getal van de vorm $n^2 + 1$ is gelijk aan 2 of congruent met 1 modulo 4, en elke priemdeler van een getal van de vorm $n^2 - 2$ is gelijk aan 2 of congruent met ± 1 modulo 8.

Voorbeelden

Voorbeeld 1. Bestaat er een rij van 2011 opeenvolgende natuurlijke getallen zodanig dat elk van deze getallen deelbaar is door de 2011-de macht van een natuurlijk getal?

Oplossing. Het antwoord is ja. Zijn $2 = p_1 < p_2 < \cdots < p_{2011}$ de eerste 2011 priemgetallen. Volgens de Chinese reststelling bestaat er een natuurlijk getal n zodanig dat $n \equiv -1 \pmod{p_1^{2011}}$, $n \equiv -2 \pmod{p_2^{2011}}$, $\cdots$, $n \equiv -2011 \pmod{p_{2011}^{2011}}$. Dan is $n + 1, n + 2, \cdots, n + 2011$ de gevraagde rij. □

Voorbeeld 2. Bepaal alle oplossingen (in gehele getallen) van de vergelijking $x^2 = y^5 - 4$.

Oplossing. Een klein beetje rekenwerk leert ons dat een kwadraat steeds congruent is met 0, 1, 3, 4, 5 of 9 modulo 11, en dat een vijfdemacht steeds congruent is met -1, 0 of 1 modulo 11.[1] Dus $y^5 - 4$ is steeds congruent met 6, 7 of 8 modulo 11. Daaruit volgt dat er geen oplossingen zijn. □

Voorbeeld 3. Bepaal alle natuurlijke getallen x, y en z zodat $3^x + 4^y = 5^z$.

[1] Trucje voor de vijfdemachten: als y niet deelbaar is door 11, dan is $y^{10} \equiv 1 \pmod{11}$, dus $11 \mid y^{10} - 1 = (y^5 - 1)(y^5 + 1)$, dus $y^5 \equiv \pm 1 \pmod{11}$.

Oplossing. Modulo 4 wordt de vergelijking $(-1)^x \equiv 1 \pmod 4$. Bijgevolg is x even. Modulo 3 wordt de vergelijking $1 \equiv (-1)^z \pmod 3$. Bijgevolg is ook z even. Dus $4^y = 2^{2y} = (5^{z/2}-3^{x/2})(5^{z/2}+3^{x/2})$. Bijgevolg geldt $5^{z/2}-3^{x/2} = 2^k$ en $5^{z/2}+3^{x/2} = 2^\ell$ met $k < \ell$ en $k+\ell = 2y$. Dus $2{\cdot}5^{z/2} = 2^k+2^\ell = 2^k(1+2^\ell)$. Daaruit volgt dat $k = 1$. Dus $5^{z/2} = 1+2^{\ell-1}$ en $3^{x/2} = -1 + 2^{\ell-1}$. Nu geldt dat machten van 3 steeds congruent zijn met 1 of 3 modulo 8 - het rechterlid van de laatste gelijkheid is echter congruent met -1 modulo 8 tenzij $\ell \leq 3$. Nu geeft $\ell = 3$ dat $x = y = z = 2$, en $\ell = 2$ geeft geen oplossing. Bijgevolg is de enige oplossing $(x, y, z) = (2, 2, 2)$.

Voorbeeld 4. (IMO 1999) Bepaal alle paren (n, p) van natuurlijke getallen n en p waarvoor geldt: p is een priemgetal, $n < 2p$ en $(p-1)^n + 1$ is deelbaar door n^{p-1}.

Oplossing. Zij q de kleinste priemdeler van n. Dan is $q \mid n^{p-1} \mid (p-1)^n + 1$, dus $(p-1)^n \equiv -1 \pmod q$. Bijgevolg is $p-1$ niet deelbaar door q. Zij α de orde van $p-1$ modulo q. Dan is $\alpha \mid q-1$ omdat $(p-1)^{q-1} \equiv 1 \pmod q$ (Fermat). Verder is ook $(p-1)^{2n} \equiv 1 \pmod q$, dus $\alpha \mid 2n$. Dus $\alpha \mid \text{ggd}(q-1, 2n)$. Maar elke deler van $q-1$ is kleiner dan q, en dus geen deler van n - want q is de *kleinste* priemdeler van n. Dus $\alpha = 1$ (als n even is) of $\alpha = 2$ (als n oneven is). Als $\alpha = 1$, dan is $q = 2$ (want n is even), maar dan moet natuurlijk ook $p = 2$, en dus $n = 2$. Als $\alpha = 2$, dan is $(p-1)^2 \equiv 1 \pmod q$, dus $q \mid p(p-2)$. Maar $p \not\equiv 2 \pmod q$ (anders zou $\alpha = 1$), dus $q \mid p$ en $q = p$. Omdat $n < 2p$ volgt daaruit dat $n = p$. Het is duidelijk dat $n = p = 3$ ook een oplossing is. Veronderstel nu dat $p \geq 5$. Dan geldt dat p^{p-1} - en dus ook p^3 - een deler is van $(p-1)^p + 1$. Maar met het binomium van Newton zien we dat deze uitdrukking gelijk is aan p^2 modulo p^3, contradictie! □

Voorbeeld 5. Zijn m en n gehele getallen. Bewijs dat $4mn - m - n$ geen volkomen kwadraat is.

Oplossing. Stel dat $4mn - m - n = a^2$. Dan is $4a^2 + 1 = (4m-1)(4n-1)$. Daaruit volgt dat $4a^2 + 1$ minstens één priemdeler heeft die congruent is met 3 modulo 4 - immers, niet alle priemdelers van $4m - 1$ kunnen congruent zijn met 1 modulo 4. Maar uit eigenschap (6) hierboven volgt dat dat niet kan. □

Voorbeeld 6. Zij $m \neq 0$ een veelvoud van 8. Hoeveel oplossingen (modulo m) heeft de kwadratische vergelijking $x^2 \equiv 1 \pmod m$ dan? Druk je antwoord uit in functie van het aantal priemdelers van m.

Oplossing. Zij $m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ de ontbinding van m, met dus $p_1 = 2$ en $a_1 \geq 3$. De congruentie $x^2 \equiv 1 \pmod m$ is volgens de Chinese reststelling equivalent met het stelsel van congruenties $x^2 \equiv 1 \pmod{p_i^{a_i}}$ voor $1 \leq i \leq r$. Dus $p_i^{a_i} \mid x^2 - 1 = (x-1)(x+1)$ voor alle i. Als $i \geq 2$ (m.a.w. als $p_i \geq 3$) dan volgt daaruit dat $x \equiv \pm 1 \pmod{p_i^{a_i}}$, want $p_i^{a_i}$ kan slechts één van de factoren $x - 1$ en $x + 1$ delen. Verder moet dus $2^{a_1} \mid (x-1)(x+1)$. Nu zal $\text{ggd}(x-1, x+1) = 2$, en dus moet x congruent zijn met 1, -1, $1 + 2^{a_1-1}$ of $-1 + 2^{a_1-1}$ modulo 2^{a_1}. Volgens de Chinese reststelling kunnen we nu de aantallen oplossingen modulo elke factor $p_i^{a_i}$ gewoon vermenigvuldigen om het aantal oplossingen modulo m te bekomen - denk daar even over na! - en we zien dus dat het aantal oplossingen gelijk is aan $2^{r-1} \cdot 4 = 2^{r+1}$. □

Opgaven

(1) Bepaal alle natuurlijke getallen n zodat $2^n \mid 3^n - 1$.

(2) Zij $p \geq 5$ een priemgetal. Bewijs dat $7^p - 6^p - 1$ deelbaar is door 43.

(3) Zij m een geheel getal zodat er een primitieve wortel a modulo m bestaat. Bewijs: $a^{\varphi(m)/2} \equiv -1 \pmod m$.

(4) Toon aan dat $2^{2\cdot 3^{n-1}} \equiv 1 + 3^n \pmod{3^{n+1}}$ voor alle n en dat 2 een primitieve wortel is modulo 3^n, voor $n \geq 1$.

(5) Bepaal de grootste gemene deler van alle getallen van de vorm $n^{13} - n$, voor $n \in \mathbb{Z}$.

(6) Zij $n \geq 2$ een natuurlijk getal. Bewijs dat $2^n - 1$ niet deelbaar is door n.

(7) (IMO 2006) Beschouw de rij $(a_n)_{n\geq 1}$ gegeven door $a_n = 2^n + 3^n + 6^n - 1$. Bepaal alle natuurlijke getallen die onderling ondeelbaar zijn met elke term van deze rij.

(8) Bepaal de drie laatste cijfers van het getal $2003^{2002^{2001}}$.

(9) (IMO 1976) Wanneer 4444^{4444} in decimale schrijfwijze wordt geschreven, dan is de som van de cijfers gelijk aan A. Zij B de som de cijfers van A. Wat is de som van de cijfers van B?

(10) Zij n een natuurlijk getal en zij $p = 2^n + 1$. Veronderstel dat $p \mid 3^{(p-1)/2} + 1$. Bewijs dat p dan een priemgetal is.

(11) (LIMO 2007) Zij n een natuurlijk getal, p een priemgetal en d een deler van $(n+1)^p - n^p$. Bewijs dat $d \equiv 1 \pmod p$.

(12) Zij n een natuurlijk getal en zij p een priemgetal met $p \leq n$. Bewijs dat

$$\binom{n}{p} \equiv \left\lfloor \frac{n}{p} \right\rfloor \pmod p.$$

(13) (IMO 1996) Zijn a en b natuurlijke getallen (verschillend van 0) zodat $15a + 16b$ en $16a - 15b$ volkomen kwadraten zijn. Bepaal de kleinst mogelijke waarde van het kleinste van deze twee kwadraten.

(14) Zij $n \geq 3$ een oneven getal. Beschouw de verzameling S van gehele getallen x zodat $1 \leq x \leq n$ en zodat x en $x+1$ allebei onderling ondeelbaar zijn met n. Bewijs dat het product van de elementen van S congruent is met 1 modulo n.

(15) (IMO 1990) Bepaal alle natuurlijke getallen n zodat $n^2 \mid 2^n + 1$. (*Hint: 2 een primitieve wortel is modulo* 3^ℓ.)

3 Uitsmijter: meer over kwadraatresten

Voor diegenen die de bovenstaande theorie al hebben gezien, een “uitsmijter”: kwadratische reciprociteit in een notendop... Zij p een oneven priemgetal en zij n een geheel getal. Als $p \mid n$, dan stellen we $(\frac{n}{p}) = 0$. Als p geen deler is van n, dan noteren we $(\frac{n}{p}) = 1$ indien n een kwadraat is modulo p en $(\frac{n}{p}) = -1$ als n geen kwadraat is modulo p. We noemen $(\frac{n}{p})$ het *Légendre-symbool*. Bewijs eerst zelf de volgende eigenschappen:

- Er geldt $(\frac{n}{p}) \equiv n^{(p-1)/2} \pmod p$.
- Voor alle $m, n \in \mathbb{Z}$ is $(\frac{mn}{p}) = (\frac{m}{p})(\frac{n}{p})$.
- Er geldt $(\frac{1}{p}) + (\frac{2}{p}) + (\frac{3}{p}) + \cdots + (\frac{p-1}{p}) = 0$ - er zijn dus evenveel kwadraten modulo p als niet-kwadraten modulo p.
- Er geldt $(\frac{-1}{p}) = (-1)^{(p-1)/2}$ en $(\frac{2}{p}) = (-1)^{(p^2-1)/8}$.

De volgende stelling moet je niet proberen te bewijzen:

Stelling. (Kwadratische reciprociteit) Voor oneven priemgetallen p en q geldt dat

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{1}{2}(p-1)\cdot\frac{1}{2}(q-1)}.$$

Voorbeeld 1. Zij $n \geq 3$ oneven en zij p een priemdeler van $2^n - 1$. Bewijs dat $p \equiv \pm 1 \pmod 8$.

Oplossing. Stel $n = 2m + 1$. Dan is $2 \cdot (2^m)^2 \equiv 1 \pmod p$. Daaruit volgt dat $(\frac{2}{p}) \equiv 1 \pmod p$ - immers, stel α is de inverse van 2^m modulo p, dan is $\alpha^2 \equiv 2 \pmod p$. Uit de bovenstaande eigenschappen volgt dan dat $p \equiv \pm 1 \pmod 8$. □

Voorbeeld 2. Voor welke priemgetallen p heeft de congruentie $x^2 \equiv -3 \pmod p$ een oplossing?

Oplossing. Voor $p = 2$ en $p = 3$ is er uiteraard een oplossing. Stel $p \geq 5$. We moeten alle p vinden zodat $(\frac{-3}{p}) = 1$, m.a.w. zodat $(\frac{-1}{p})(\frac{3}{p}) = 1$. Maar $(\frac{3}{p}) = (\frac{p}{3}) \cdot (-1)^{(p-1)/2}$ wegens de kwadratische reciprociteitswet, en $(\frac{-1}{p}) = (-1)^{(p-1)/2}$, dus we besluiten dat $(\frac{-3}{p}) = (\frac{p}{3})(-1)^{p-1} = (\frac{p}{3})$. Bijgevolg voldoen alle p met $p \equiv 1 \pmod 3$. □

Voorbeeld 3. Voor welke natuurlijke getallen n bestaat er een natuurlijk getal m zodat $2^n - 1 \mid m^2 + 9$?

Oplossing. We bewijzen dat m bestaat als en slechts als $n = 2^k$ (met $k \geq 0$). Stel eerst dat n geen macht van 2 is. Kies dus een oneven priemdeler p van n: dan geldt $2^p - 1 \mid m^2 + 9$. Kies een priemfactor q van $2^p - 1$ met $q \equiv 3 \pmod 4$ en $q \neq 3$ (ga na dat q bestaat!). Dan $1 = (\frac{-9}{q}) = (\frac{9}{q})(\frac{-1}{q}) = (\frac{-1}{q}) = -1$ (omdat $q \equiv 3 \pmod 4$), contradictie. Bijgevolg bestaat m niet. Stel nu dat $n = 2^k$. Stel $T_r = 2^{2^r} + 1$, dan is $2^n - 1 = T_0 T_1 T_2 \cdots T_{k-1}$. Merk nu op dat $\text{ggd}(T_i, T_j) = 1$ als $i \neq j$ (ga dat na als oefening!). Kies een natuurlijk getal m zodat $m \equiv 2 \pmod{T_1}$, $m \equiv 2^2 \pmod{T_2}, \ldots, m \equiv 2^{2^{k-2}} \pmod{T_{k-1}}$ (Chinese reststelling). Dan is $m^2 + 1$ deelbaar door $T_1 T_2 \cdots T_{k-1}$, en dus is $(3m)^2 + 9$ deelbaar door $2^n - 1$. □

Opgaven

(1) Bewijs dat 16 een volkomen achtste macht is modulo p, voor elk priemgetal p.

(2) Zijn a, b, c paarsgewijs onderling ondeelbare natuurlijke getallen met $c^2 = a^2 - ab + b^2$. Zij p een priemdeler van c. Bewijs dat $p \equiv 1 \pmod 6$.

(3) Zij F_n het n-de Fibonacci-getal. Bewijs dat voor elk priemgetal $p \geq 7$ geldt dat $F_p \equiv (\frac{p}{5}) \pmod p$.

4 Meer oefenmateriaal!

Hierna volgt nog een lijst van 35 leuke problemen die kan dienen als extra oefenmateriaal. Voor sommige opgaven zal de theorie die hierboven werd aangehaald erg nuttig zijn, maar er zitten ook opgaven bij die niet rechtstreeks aansluiten op de theorie. Ik heb de opgaven gerangschikt op moeilijkheidsgraad, maar die rangschikking is natuurlijk subjectief. . .

(1) Drie Amerikaanse wiskundigen gaven een tegenvoorbeeld voor een bekend vermoeden van Euler (in de jaren 1980) door aan te tonen dat er een natuurlijk getal n bestaat zodat $n^5 = 133^5 + 110^5 + 84^5 + 27^5$. Wat is de waarde van n?

(2) Het getal 2198214591730833048701336 9 is de dertiende macht van een natuurlijk getal. Welk getal?

(3) Stel $34! = 95232799cd96041408476186096435ab000000$. Bepaal de cijfers a, b, c en d.

(4) Laat zien dat de vergelijking $x^2 + y^5 = z^3$ oneindig veel gehele oplossingen heeft met $x, y, z \neq 0$.

(5) Zijn a en b natuurlijke getallen zodat $2^n a + b$ een volkomen kwadraat is voor alle natuurlijke getallen n. Bewijs: $a = 0$.

(6) Toon aan dat oneindig veel natuurlijke getallen niet kunnen worden geschreven als $x^2 + y^3 + z^7$, met $x, y, z \in \mathbb{N}$.

(7) Zij n een natuurlijk getal zodat $N = 2 + 2\sqrt{28n^2 + 1}$ een natuurlijk getal is. Bewijs dat N een volkomen kwadraat is.

(8) Bepaal alle $a, b \in \mathbb{N}$ zodat $(a + 19b)^{18} + (a + b)^{18} + (19a + b)^{18}$ een volkomen kwadraat is.

(9) Bewijs dat voor alle natuurlijke getallen n geldt: $7 \mid n^3 + 3^n \iff 7 \mid n^3 3^n + 1$.

(10) Definieer voor elk natuurlijk getal n het getal $p(n)$ als de grootste oneven deler van n. Bewijs:

$$\frac{1}{2^k} \sum_{n=1}^{2^k} \frac{p(n)}{n} > \frac{2}{3}.$$

(11) (IMO 1986) Zij d een natuurlijk getal met $d \notin \{0, 2, 5, 13\}$. Bewijs dat er in de verzameling $\{2, 5, 13, d\}$ steeds twee getallen a en b zitten zodanig dat het getal $ab - 1$ geen volkomen kwadraat is.

(12) Zijn n en q natuurlijke getallen met $n \geq 5$ en $2 \leq q \leq n$. Bewijs dat $q - 1$ een deler is van $\lfloor \frac{(n-1)!}{q} \rfloor$.

(13) Bepaal alle natuurlijke oplossingen van de vergelijking $a! \cdot b! = a! + b! + c!$.

(14) Toon aan dat elk geheel getal de som is van vijf volkomen derdemachten.

(15) We noemen $n \in \mathbb{N}$ *machtig* als n de volgende eigenschap heeft: als $n \mid a^n - 1$ voor een zekere $a \in \mathbb{N}$, dan geldt $n^2 \mid a^n - 1$. Bewijs dat priemgetallen machtig zijn, en dat oneindig veel niet-priemgetallen machtig zijn.

(16) Bepaal alle natuurlijke getallen a en b zodat $(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 49 + 20\sqrt[3]{6}$.

(17) Bepaal alle rekenkundige rijtjes van drie natuurlijke getallen met de eigenschap dat het product van de drie termen van het rijtje geen priemfactor heeft die strikt groter is dan 3.

(18) Zij α de grootste wortel van de vergelijking $x^3 - 3x^2 + 1 = 0$. Bewijs dat $\lfloor \alpha^{1788} \rfloor$ en $\lfloor \alpha^{1988} \rfloor$ deelbaar zijn door 17.

(19) Bewijs dat voor elk natuurlijk getal n geldt dat $\lfloor \sqrt[3]{n} + \sqrt[3]{n+1} \rfloor = \lfloor \sqrt[3]{8n+3} \rfloor$.

(20) Zij N een natuurlijk getal. Bewijs dat er een rij van N *opeenvolgende* natuurlijke getallen bestaat zodanig dat de j-de term van deze rij de som is van j *verschillende* volkomen kwadraten.

(21) Definieer $a_0 = 0$, $a_1 = 1$ en $a_{n+2} = 2a_{n+1} + a_n$ voor $n \geq 0$. Bewijs: $2^k \mid a_n \iff 2^k \mid n$.

(22) Defininieer een rij van natuurlijke getallen door $u_0 = 1$ en $u_{n+1} = au_n + b$, waarbij a en b willekeurige natuurlijke getallen zijn. Bewijs dat deze rij (voor elke keuze van a en b) oneindig veel termen heeft die niet priem zijn.

(23) Bewijs dat er oneindig veel natuurlijke getallen n bestaan met $n^2 + 1 \mid n!$.

(24) Definieer $y_0 = 1$ en $y_{n+1} = \frac{1}{2}(3y_n + \sqrt{5y_n^2 - 4})$. Bewijs dat $y_n \in \mathbb{N}$ voor alle n.

(25) (IMO 2006) Bepaal alle natuurlijke getallen x en y zodat $y^2 = 1 + 2^x + 2^{2x+1}$.

(26) (IMO 1997) Bepaal alle gehele getallen a en b zodat $a^{b^2} = b^a$.

(27) (IMO 2003) Bepaal alle gehele getallen a en b zodat $a^2/(2ab^2 - b^3 + 1)$ een natuurlijk getal is.

(28) Een deelnemer aan het IMO-stageweekend die niet goed heeft opgelet herinnert zich de kleine stelling van Fermat als volgt: als p een priemgetal is en a een natuurlijk getal, dan is $a^{p+1} \equiv a \pmod{p}$. Dat slaat natuurlijk nergens op, en iedereen die opgelet heeft weet dat de juiste congruentie de volgende is: $a^p \equiv a \pmod{p}$. Maar toch de volgende vraag: welke natuurlijke getallen p hebben de eigenschap dat $a^{p+1} \equiv a \pmod{p}$ voor alle natuurlijke getallen a?

(29) Zijn $p_1, p_2, \cdots, p_k \geq 5$ verschillende priemgetallen. Bewijs dat $\tau(2^{p_1 p_2 \cdots p_k} + 1) \geq 4^k$.

(30) Zijn x en y natuurlijke getallen zodat xy een deler is van $x^2 + y^2 + 1$. Bewijs dat $x^2 + y^2 + 1 = 3xy$.

(31) (IMO 2007) Zijn a en b natuurlijke getallen zodat $4ab - 1 \mid (4a^2 - 1)^2$. Bewijs dat $a = b$.

(32) Bestaat er een natuurlijk getal m zodanig dat de vergelijking

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

oneindig veel natuurlijke oplossingen heeft?

(33) (IMO 1988) Zijn a en b natuurlijke getallen zodat $ab + 1$ een deler is van $a^2 + b^2$. Bewijs dat

$$\frac{a^2 + b^2}{ab + 1}$$

een volkomen kwadraat is.

(34) (IMO 2000) Bestaat er een natuurlijk getal n met precies 2000 priemdelers zodat n een deler is van $2^n + 1$?

(35) (IMO 1998) Voor welke natuurlijke getallen k bestaat er een natuurlijk getal k zodat $\tau(n^2) = k\tau(n)$?

terug naar echt bestand

6.2 lifting the exponent lemma

Lifting The Exponent Lemma (LTE)

Version 6 - Amir Hossein Parvardi

April 7, 2011

Lifting The Exponent Lemma is a powerful method for solving exponential Diophantine equations. It is pretty well-known in the Olympiad folklore (see, e.g., [3]) though its origins are hard to trace. Mathematically, it is a close relative of the classical Hensel's lemma (see [2]) in number theory (in both the statement and the idea of the proof). In this article we analyze this method and present some of its applications.

We can use the Lifting The Exponent Lemma (this is a long name, let's call it **LTE**!) in lots of problems involving exponential equations, especially when we have some prime numbers (and actually in some cases it "explodes" the problems). This lemma shows how to find the greatest power of a prime p – which is often ≥ 3 – that divides $a^n \pm b^n$ for some positive integers a and b. The proofs of theorems and lemmas in this article have nothing difficult and all of them use elementary mathematics. Understanding the theorem's usage and its meaning is more important to you than remembering its detailed proof.

I have to thank Fedja, darij grinberg(Darij Grinberg), makar and ZetaX(Daniel) for their notifications about the article. And I specially appreciate JBL(Joel) and Fedja helps about TeX issues.

1 Definitions and Notation

For two integers a and b we say a is divisible by b and write $b \mid a$ if and only if there exists some integer q such that $a = qb$.

We define $v_p(x)$ to be the greatest power in which a prime p divides x; in particular, if $v_p(x) = \alpha$ then $p^\alpha \mid x$ but $p^{\alpha+1} \nmid x$. We also write $p^\alpha \| x$, if and only if $v_p(x) = \alpha$. So we have $v_p(xy) = v_p(x) + v_p(y)$ and $v_p(x + y) \geq \min \{v_p(x), v_p(y)\}$.

Example. The greatest power of 3 that divides 63 is 3^2. because $3^2 = 9 \mid 63$ but $3^3 = 27 \nmid 63$. in particular, $3^2 \| 63$ or $v_3(63) = 2$.

Example. Clearly we see that if p and q are two different prime numbers, then $v_p(p^\alpha q^\beta) = \alpha$, or $p^\alpha \| p^\alpha q^\beta$.

Note. We have $v_p(0) = \infty$ for all primes p.

2 Two Important and Useful Lemmas

Lemma 1. *Let x and y be (not necessary positive) integers and let n be a positive integer. Given an arbitrary prime p (in particular, we can have $p = 2$) such that $\gcd(n, p) = 1$, $p \mid x - y$ and neither x, nor y is divisible by p (i.e., $p \nmid x$ and $p \nmid y$). We have*

$$v_p(x^n - y^n) = v_p(x - y).$$

Proof. We use the fact that

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}).$$

Now if we show that $p \nmid x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}$, then we are done. In order to show this, we use the assumption $p \mid x - y$. So we have $x - y \equiv 0 \pmod{p}$, or $x \equiv y \pmod{p}$. Thus

$$\begin{aligned} x^{n-1} + & x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1} \\ & \equiv x^{n-1} + x^{n-2} \cdot x + x^{n-3} \cdot x^2 + \cdots + x \cdot x^{n-2} + x^{n-1} \\ & \equiv nx^{n-1} \\ & \not\equiv 0 \pmod{p}. \end{aligned}$$

This completes the proof. □

Lemma 2. *Let x and y be (not necessary positive) integers and let n be an odd positive integer. Given an arbitrary prime p (in particular, we can have $p = 2$) such that $\gcd(n, p) = 1$, $p \mid x + y$ and neither x, nor y is divisible by p, we have*

$$v_p(x^n + y^n) = v_p(x + y).$$

Proof. Since x and y can be negative, using **Lemma 1** we obtain

$$v_p(x^n - (-y)^n) = v_p(x - (-y)) \implies v_p(x^n + y^n) = v_p(x + y).$$

Note that since n is an odd positive integer we can replace $(-y)^n$ with $-y^n$. □

3 Lifting The Exponent Lemma (LTE)

Theorem 1 (First Form of LTE)**.** *Let x and y be (not necessary positive) integers, let n be a positive integer, and let p be an odd prime such that $p \mid x - y$ and none of x and y is divisible by p (i.e., $p \nmid x$ and $p \nmid y$). We have*

$$v_p(x^n - y^n) = v_p(x - y) + v_p(n).$$

Proof. We may use induction on $v_p(n)$. First, let us prove the following statement:

$$v_p(x^p - y^p) = v_p(x - y) + 1. \tag{1}$$

In order to prove this, we will show that

$$p \mid x^{p-1} + x^{p-2}y + \cdots + xy^{p-2} + y^{p-1} \tag{2}$$

and

$$p^2 \nmid x^{p-1} + x^{p-2}y + \cdots + xy^{p-2} + y^{p-1}. \tag{3}$$

For **(2)**, we note that

$$x^{p-1} + x^{p-2}y + \cdots + xy^{p-2} + y^{p-1} \equiv px^{p-1} \equiv 0 \pmod{p}.$$

Now, let $y = x + kp$, where k is an integer. For an integer $1 \le t < p$ we have

$$\begin{aligned} y^t x^{p-1-t} &\equiv (x+kp)^t x^{p-1-t} \\ &\equiv x^{p-1-t}\left(x^t + t(kp)(x^{t-1}) + \frac{t(t-1)}{2}(kp)^2(x^{t-2}) + \cdots\right) \\ &\equiv x^{p-1-t}\left(x^t + t(kp)(x^{t-1})\right) \\ &\equiv x^{p-1} + tkpx^{p-2} \pmod{p^2}. \end{aligned}$$

This means

$$y^t x^{p-1-t} \equiv x^{p-1} + tkpx^{p-2} \pmod{p^2}, \quad t = 1, 2, 3, 4, \ldots, p-1.$$

Using this fact, we have

$$\begin{aligned} &x^{p-1} + x^{p-2}y + \cdots + xy^{p-2} + y^{p-1} \\ &\quad\equiv x^{p-1} + (x^{p-1} + kpx^{p-2}) + (x^{p-1} + 2kpx^{p-2}) + \cdots + (x^{p-1} + (p-1)kpx^{p-2}) \\ &\quad\equiv px^{p-1} + (1 + 2 + \cdots + p - 1)kpx^{p-2} \\ &\quad\equiv px^{p-1} + \left(\frac{p(p-1)}{2}\right)kpx^{p-2} \\ &\quad\equiv px^{p-1} + \left(\frac{p-1}{2}\right)kp^2x^{p-1} \\ &\quad\equiv px^{p-1} \not\equiv 0 \pmod{p^2}. \end{aligned}$$

So we proved **(3)** and the proof of **(1)** is complete. Now let us return to our problem. We want to show that

$$v_p(x^n - y^n) = v_p(x-y) + v_p(n).$$

Suppose that $n = p^\alpha b$ where $\gcd(p, b) = 1$. Then

$$\begin{aligned} v_p(x^n - y^n) &= v_p((x^{p^\alpha})^b - (y^{p^\alpha})^b) \\ &= v_p(x^{p^\alpha} - y^{p^\alpha}) = v_p((x^{p^{\alpha-1}})^p - (y^{p^{\alpha-1}})^p) \\ &= v_p(x^{p^{\alpha-1}} - y^{p^{\alpha-1}}) + 1 = v_p((x^{p^{\alpha-2}})^p - (y^{p^{\alpha-2}})^p) + 1 \\ &= v_p(x^{p^{\alpha-2}} - y^{p^{\alpha-2}}) + 2 \\ &\ \ \vdots \\ &= v_p((x^{p^1})^1 - (y^{p^1})^1) + \alpha - 1 = v_p(x-y) + \alpha \\ &= v_p(x-y) + v_p(n). \end{aligned}$$

Note that we used the fact that if $p \mid x-y$, then we have $p \mid x^k - y^k$, because we have $x - y \mid x^k - y^k$ for all positive integers k. The proof is complete. □

Theorem 2 (Second Form of LTE)**.** *Let x, y be two integers, n be an odd positive integer, and p be an odd prime such that $p \mid x + y$ and none of x and y is divisible by p. We have*

$$v_p(x^n + y^n) = v_p(x + y) + v_p(n).$$

Proof. This is obvious using **Theorem 1**. See the trick we used in proof of **Lemma 2**. □

4 What about $p = 2$?

Question. Why did we assume that p is an odd prime, i.e., $p \neq 2$? Why can't we assume that $p = 2$ in our proofs?

Hint. Note that $\frac{p-1}{2}$ is an integer only for $p > 2$.

Theorem 3 (LTE for the case $p = 2$)**.** *Let x and y be two odd integers such that $4 \mid x - y$. Then*

$$v_2(x^n - y^n) = v_2(x - y) + v_2(n).$$

Proof. We showed that for any prime p such that $\gcd(p, n) = 1, p \mid x - y$ and none of x and y is divisible by p, we have

$$v_p(x^n - y^n) = v_p(x - y)$$

So it suffices to show that

$$v_2(x^{2^n} - y^{2^n}) = v_2(x - y) + n.$$

Factorization gives

$$x^{2^n} - y^{2^n} = (x^{2^{n-1}} + y^{2^{n-1}})(x^{2^{n-2}} + y^{2^{n-2}}) \cdots (x^2 + y^2)(x + y)(x - y)$$

Now since $x \equiv y \equiv \pm 1 \pmod 4$ then we have $x^{2^k} \equiv y^{2^k} \equiv 1 \pmod 4$ for all positive integers k and so $x^{2^k} + y^{2^k} \equiv 2 \pmod 4), k = 1, 2, 3, \ldots$. Also, since x and y are odd and $4 \mid x - y$, we have $x + y \equiv 2 \pmod 4$. This means the power of 2 in all of the factors in the above product (except $x - y$) is one. We are done. □

Theorem 4. *Let x and y be two odd integers and let n be an even positive integer. Then*

$$v_2(x^n - y^n) = v_2(x - y) + v_2(x + y) + v_2(n) - 1.$$

Proof. We know that the square of an odd integer is of the form $4k+1$. So for odd x and y we have $4 \mid x^2 - y^2$. Now let m be an odd integer and k be a positive integer such that $n = m \cdot 2^k$. Then

$$\begin{aligned} v_2(x^n - y^n) &= v_2(x^{m \cdot 2^k} - y^{m \cdot 2^k}) \\ &= v_2((x^2)^{2^{k-1}} - (y^2)^{2^{k-1}}) \\ &\vdots \\ &= v_2(x^2 - y^2) + k - 1 \\ &= v_2(x-y) + v_2(x+y) + v_2(n) - 1. \end{aligned}$$

□

5 Summary

Let p be a prime number and let x and y be two (not necessary positive) integers that are not divisible by p. Then:

a) For a positive integer n

- if $p \neq 2$ and $p \mid x - y$, then

$$v_p(x^n - y^n) = v_p(x-y) + v_p(n).$$

- if $p = 2$ and $4 \mid x - y$, then

$$v_2(x^n - y^n) = v_2(x-y) + v_2(n).$$

- if $p = 2$, n is even, and $2 \mid x - y$, then

$$v_2(x^n - y^n) = v_2(x-y) + v_2(x+y) + v_2(n) - 1.$$

b) For an odd positive integer n, if $p \mid x + y$, then

$$v_p(x^n + y^n) = v_p(x+y) + v_p(n).$$

c) For a positive integer n with $\gcd(p, n) = 1$, if $p \mid x - y$, we have

$$v_p(x^n - y^n) = v_p(x-y).$$

If n is odd, $\gcd(p, n) = 1$, and $p \mid x + y$, then we have

$$v_p(x^n + y^n) = v_p(x+y).$$

Note. The most common mistake in using LTE is when you don't check the $p \mid x \pm y$ condition, so always remember to check it. Otherwise your solution will be completely wrong.

6 Problems with Solutions

Problem 1 (Russia 1996)**.** *Find all positive integers n for which there exist positive integers x, y and k such that* $\gcd(x, y) = 1, k > 1$ *and* $3^n = x^k + y^k$.

Solution. k should be an odd integer (otherwise, if k is even, then x^k and y^k are perfect squares, and it is well known that for integers a, b we have $3 \mid a^2 + b^2$ if and only if $3 \mid a$ and $3 \mid b$, which is in contradiction with $\gcd(x, y) = 1$.). Suppose that there exists a prime p such that $p \mid x + y$. This prime should be odd. So $v_p(3^n) = v_p(x^k + y^k)$, and using **Theorem 2** we have $v_p(3^n) = v_p(x^k + y^k) = v_p(k) + v_p(x + y)$. But $p \mid x + y$ means that $v_p(x + y) \geq 1 > 0$ and so $v_p(3^n) = v_p(k) + v_p(x + y) > 0$ and so $p \mid 3^n$. Thus $p = 3$. This means $x + y = 3^m$ for some positive integer m. Note that $n = v_3(k) + m$. There are two cases:

- $m > 1$. We can prove by induction that $3^a \geq a + 2$ for all integers $a \geq 1$, and so we have $v_3(k) \leq k - 2$ (why?). Let $M = \max(x, y)$. Since $x + y = 3^m \geq 9$, we have $M \geq 5$. Then

$$x^k + y^k \geq M^k = \underbrace{M}_{\geq \frac{x+y}{2} = \frac{1}{2} \cdot 3^m} \cdot \underbrace{M^{k-1}}_{\geq 5^{k-1}} > \frac{1}{2} 3^m \cdot 5^{k-1}$$
$$> 3^m \cdot 5^{k-2} \geq 3^{m+k-2} \geq 3^{m+v_3(k)} = 3^n$$

  which is a contradiction.

- $m = 1$. Then $x + y = 3$, so $x = 1, y = 2$ (or $x = 2, y = 1$). Thus $3^{1+v_3(k)} = 1 + 2^k$. But note that $3^{v_3(k)} \mid k$ so $3^{v_3(k)} \leq k$. Thus

$$1 + 2^k = 3^{v_3(k)+1} = 3 \cdot \underbrace{3^{v_3(k)}}_{\leq k} \leq 3k \implies 2^k + 1 \leq 3k.$$

  And one can check that the only odd value of $k > 1$ that satisfies the above inequality is $k = 3$. So $(x, y, n, k) = (1, 2, 2, 3), (2, 1, 2, 3)$ in this case.

Thus, the final answer is $n = 2$.

Problem 2 (Balkan 1993)**.** *Let p be a prime number and $m > 1$ be a positive integer. Show that if for some positive integers $x > 1, y > 1$ we have*

$$\frac{x^p + y^p}{2} = \left(\frac{x + y}{2}\right)^m,$$

then $m = p$.

Solution. One can prove by induction on p that $\frac{x^p+y^p}{2} \geq \left(\frac{x+y}{2}\right)^p$ for all positive integers p. Now since $\frac{x^p+y^p}{2} = \left(\frac{x+y}{2}\right)^m$, we should have $m \geq p$. Let $d = \gcd(x, y)$, so there exist positive integers x_1, y_1 with $\gcd(x_1, y_1) = 1$ such that $x = dx_1$, $y = dy_1$ and $2^{m-1}(x_1^p + y_1^p) = d^{m-p}(x_1 + y_1)^m$. There are two cases:

Assume that p is odd. Take any prime divisor q of x_1+y_1 and let $v = v_q(x_1+y_1)$. If q is odd, we see that $v_q(x_1^p+y_1^p) = v+v_q(p)$ and $v_q(d^{m-p}(x_1+y_1)^m) \geq mv$ (because q may also be a factor of d). Thus $m \leq 2$ and $p \leq 2$, giving an immediate contradiction. If $q = 2$, then $m-1+v \geq mv$, so $v \leq 1$ and $x_1+y_1 = 2$, i.e., $x = y$, which immediately implies $m = p$.

Assume that $p = 2$. We notice that for $x+y \geq 4$ we have $\frac{x^2+y^2}{2} < 2\left(\frac{x+y}{2}\right)^2 \leq \left(\frac{x+y}{2}\right)^3$, so $m = 2$. It remains to check that the remaining cases $(x,y) = (1,2),(2,1)$ are impossible.

Problem 3. *Find all positive integers a, b that are greater than 1 and satisfy*

$$b^a | a^b - 1.$$

Solution. Let p be the least prime divisor of b. Let m be the least positive integer for which $p|a^m-1$. Then $m|b$ and $m \mid p-1$, so any prime divisor of m divides b and is less than p. Thus, not to run into a contradiction, we must have $m = 1$. Now, if p is odd, we have $av_p(b) \leq v_p(a-1)+v_p(b)$, so $a-1 \leq (a-1)v_p(b) \leq v_p(a-1)$, which is impossible. Thus $p = 2$, b is even, a is odd and $av_2(b) \leq v_2(a-1)+v_2(a+1)+v_2(b)-1$ whence $a \leq (a-1)v_2(b)+1 \leq v_2(a-1)+v_2(a+1)$, which is possible only if $a = 3$, $v_2(b) = 1$. Put $b = 2B$ with odd B and rewrite the condition as $2^3B^3 \mid 3^{2B}-1$. Let q be the least prime divisor of B (now, surely, odd). Let n be the least positive integer such that $q \mid 3^n-1$. Then $n \mid 2B$ and $n \mid q-1$ whence n must be 1 or 2 (or B has a smaller prime divisor), so $q \mid 3-1 = 2$ or $q \mid 3^2-1 = 8$, which is impossible. Thus $B = 1$ and $b = 2$.

Problem 4. *Find all positive integer solutions of the equation $x^{2009}+y^{2009} = 7^z$*

Solution. Factor 2009. We have $2009 = 7^2 \cdot 41$. Since $x+y \mid x^{2009}+y^{2009}$ and $x+y > 1$, we must have $7 \mid x+y$. Removing the highest possible power of 7 from x, y, we get $v_7(x^{2009}+y^{2009}) = v_7(x+y)+v_7(2009) = v_7(x+y)+2$, so $x^{2009}+y^{2009} = 49 \cdot k \cdot (x+y)$ where $7 \nmid k$. But we have $x^{2009}+y^{2009} = 7^z$, which means the only prime factor of $x^{2009}+y^{2009}$ is 7, so $k = 1$). Thus $x^{2009}+y^{2009} = 49(x+y)$. But in this equation the left hand side is much larger than the right hand one if $\max(x,y) > 1$, and, clearly, $(x,y) = (1,1)$ is not a solution. Thus the given equation does not have any solutions in the set of positive integers.

7 Challenge Problems

1. Let k be a positive integer. Find all positive integers n such that $3^k \mid 2^n - 1$.

2 (UNESCO Competition 1995)**.** Let a, n be two positive integers and let p be an odd prime number such that

$$a^p \equiv 1 \pmod{p^n}.$$

Prove that

$$a \equiv 1 \pmod{p^{n-1}}.$$

3 (Iran Second Round 2008)**.** Show that the only positive integer value of a for which $4(a^n + 1)$ is a perfect cube for all positive integers n, is 1.

4. Let $k > 1$ be an integer. Show that there exists infinitely many positive integers n such that

$$n | 1^n + 2^n + 3^n + \cdots + k^n.$$

5 (Ireland 1996)**.** Let p be a prime number, and a and n positive integers. Prove that if

$$2^p + 3^p = a^n$$

then $n = 1$.

6 (Russia 1996)**.** Let x, y, p, n, k be positive integers such that n is odd and p is an odd prime. Prove that if $x^n + y^n = p^k$, then n is a power of p.

7. Find the sum of all the divisors d of $N = 19^{88} - 1$ which are of the form $d = 2^a 3^b$ with $a, b \in \mathbb{N}$.

8. Let p be a prime number. Solve the equation $a^p - 1 = p^k$ in the set of positive integers.

9. Find all solutions of the equation

$$(n-1)! + 1 = n^m$$

in positive integers.

10 (Bulgaria 1997)**.** For some positive integer n, the number $3^n - 2^n$ is a perfect power of a prime. Prove that n is a prime.

11. Let m, n, b be three positive integers with $m \neq n$ and $b > 1$. Show that if prime divisors of the numbers $b^n - 1$ and $b^m - 1$ be the same, then $b + 1$ is a perfect power of 2.

12 (IMO ShortList 1991)**.** Find the highest degree k of 1991 for which 1991^k divides the number

$$1990^{1991^{1992}} + 1992^{1991^{1990}}.$$

13. Prove that the number $a^{a-1} - 1$ is never square-free for all integers $a > 2$.

14 (Czech Slovakia 1996)**.** Find all positive integers x, y such that $p^x - y^p = 1$, where p is a prime.

15. Let x and y be two positive rational numbers such that for infinitely many positive integers n, the number $x^n - y^n$ is a positive integer. Show that x and y are both positive integers.

16 (IMO 2000)**.** Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides $2^n + 1$?

17 (China Western Mathematical Olympiad 2010)**.** Suppose that m and k are non-negative integers, and $p = 2^{2^m} + 1$ is a prime number. Prove that

- $2^{2^{m+1}p^k} \equiv 1 \pmod{p^{k+1}}$;
- $2^{m+1}p^k$ is the smallest positive integer n satisfying the congruence equation $2^n \equiv 1 \pmod{p^{k+1}}$.

18. Let $p \geq 5$ be a prime. Find the maximum value of positive integer k such that

$$p^k|(p-2)^{2(p-1)} - (p-4)^{p-1}.$$

19. Let a, b be distinct real numbers such that the numbers

$$a - b,\ a^2 - b^2,\ a^3 - b^3, \ldots$$

are all integers. Prove that a, b are both integers.

20 (MOSP 2001)**.** Find all quadruples of positive integers (x, r, p, n) such that p is a prime number, $n, r > 1$ and $x^r - 1 = p^n$.

21 (China TST 2009)**.** Let $a > b > 1$ be positive integers and b be an odd number, let n be a positive integer. If $b^n \mid a^n - 1$, then show that $a^b > \frac{3^n}{n}$.

22 (Romanian Junior Balkan TST 2008)**.** Let p be a prime number, $p \neq 3$, and integers a, b such that $p \mid a + b$ and $p^2 \mid a^3 + b^3$. Prove that $p^2 \mid a + b$ or $p^3 \mid a^3 + b^3$.

23. Let m and n be positive integers. Prove that for each odd positive integer b there are infinitely many primes p such that $p^n \equiv 1 \pmod{b^m}$ implies $b^{m-1} \mid n$.

24 (IMO 1990)**.** Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

25. Find all positive integers n such that

$$\frac{2^{n-1} + 1}{n}.$$

is an integer.

26. Find all primes p, q such that $\dfrac{(5^p - 2^p)(5^q - 2^q)}{pq}$ is an integer.

27. For some natural number n let a be the greatest natural number for which $5^n - 3^n$ is divisible by 2^a. Also let b be the greatest natural number such that $2^b \leq n$. Prove that $a \leq b + 3$.

28. Determine all sets of non-negative integers x, y and z which satisfy the equation

$$2^x + 3^y = z^2.$$

29 (IMO ShortList 2007)**.** Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime p, the number $f(m+n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p.

30 (Romania TST 1994)**.** Let n be an odd positive integer. Prove that $((n-1)^n + 1)^2$ divides $n(n-1)^{(n-1)^n+1} + n$.

31. Find all positive integers n such that $3^n - 1$ is divisible by 2^n.

32 (Romania TST 2009)**.** Let $a, n \geq 2$ be two integers, which have the following property: there exists an integer $k \geq 2$, such that n divides $(a-1)^k$. Prove that n also divides $a^{n-1} + a^{n-2} + \cdots + a + 1$.

33. Find all the positive integers a such that $\frac{5^a+1}{3^a}$ is a positive integer.

8 Hints and Answers to Selected Problems

1. Answer: $n = 2 \cdot 3^{k-1}s$ for some $s \in \mathbb{N}$.

2. Show that $v_p(a-1) = v_p(a^p - 1) - 1 \geq n - 1$.

3. If $a > 1$, a^2+1 is not a power of 2 (because it is > 2 and either 1 or 2 modulo 4). Choose some odd prime $p|a^2+1$. Now, take some $n = 2m$ with odd m and notice that $v_p(4(a^n+1)) = v_p(a^2+1) + v_p(m)$ but $v_p(m)$ can be anything we want modulo 3.

5. $2^p + 3^p$ is not a square, and use the fact that $v_5(2^p+3^p) = 1 + v_5(p) \leq 2$.

8. Consider two cases : $p = 2$ and p is an odd prime. The latter does not give any solutions.

9. $(n, m) = (2, 1)$ is a solution. In other cases, show that n is an odd prime and m is even. The other solution is $(n, m) = (5, 2)$.

12. Answer: $\max(k) = 1991$.

13. Take any odd prime p such that $p \mid a - 1$. It's clear that $p^2 \mid a^{a-1} - 1$.

14. Answer: $(p, x, y) = (2, 1, 1), (3, 2, 1)$.

18. Let $p - 1 = 2^s m$ and show that $v_p(2^{s-1}m) = 0$. The maximum of k is 1.

19. Try to prove Problem **15** first.

20. Show that $p = 2$ and r is an even positive integer.

22. If $p \mid a, p \mid b$, then $p^3 \mid a^3 + b^3$. Otherwise LTE applies and $v_p(a+b) = v_p(a^3+b^3) \geq 2$.

24. The answer is $n = 1$ or $n = 3$.

26. Answer: $(p, q) = (3, 3), (3, 13)$.

27. If n is odd, then $a = 1$. If n is even, then $a = v_2(5^n - 3^n) = v_2(5-3) + v_2(5+3) + v_2(n) - 1 = 3 + v_2(n)$. But, clearly, $b \geq v_2(n)$.

30. $n \mid (n-1)^n + 1$, so for every $p \mid (n-1)^n + 1$, we have

$$\begin{aligned} v_p((n-1)^{(n-1)^n+1} + 1) &= v_p((n-1)^n + 1) + v_p\left(\frac{(n-1)^{n+1}+1}{n}\right) \\ &= 2v_p((n-1)^n + 1) - v_p(n) \end{aligned}$$

which completes the proof.

31. $n \leq v_2(3^n - 1) \leq 3 + v_2(n)$, so $n \leq 4$.

33. a must be odd (otherwise the numerator is $2 \mod 3$). Then $a \leq v_3(5^a+1) = 1 + v_3(a)$ giving $a = 1$ as the only solution.

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terug naar echt bestand

6.3 het duivenhokprincipe

Het Duivenhokprincipe

Arne Smeets

1 Duivenhok, zn., onz. (-ken), hok waarin men duiven houdt

Dit is alles wat de “Dikke Van Dale” ons weet te vertellen over het woord *duivenhok*. Een saai begrip, nee? Zeker niet interessant genoeg om een hele lesbrief te vullen... maar wanneer we het hebben over het *duivenhokprincipe*, dan worden de zaken al direct veel interessanter, en dan weet de Van Dale plots van toeten noch blazen. Deze lesbrief gaat dus over het *duivenhokprincipe*, ook wel het *ladenprincipe van Dirichlet* genoemd. In het Frans spreekt men over *le principe des tiroirs* en in het Engels heeft men het over *the pigeonhole principle*, dat we zullen afkorten als “PHP”. Met behulp van een drietal eenvoudige stellingen (die we PHP1, PHP2 en PHP3 zullen noemen) kunnen we een aantal zeer leuke en uiteenlopende problemen oplossen: hoofdzakelijk opgaven uit de combinatoriek, maar soms ook algebraïsche, getaltheoretische en zelfs meetkundige problemen. Vaak zijn dit moeilijke vragen: het duivenhokprincipe mag dan wel zeer eenvoudig zijn, in veel gevallen is het verre van evident om het op de juiste manier toe te passen. Je bent gewaarschuwd... Succes!

2 Lang geleden waren er eens n duiven en k duivenhokken...

Veel theorie valt er hier niet te bespreken: ik presenteer hier drie eenvoudige stellingen, die door sommigen misschien zelfs als “vanzelfsprekend” en logisch zullen worden beschouwd. Men gebruikt dikwijls *duiven* en *duivenhokken* (of *laden* en *voorwerpen*) om deze stellingen te formuleren, maar in feite handelen deze stellingen over verzamelingen en functies. Na elke stelling vermeld ik dus ook de “formele”(wiskundig correcte) versie van de stelling. We zullen in dit hoofdstukje het aantal elementen van een verzameling S voorstellen door $|S|$. Met $[x]$ bedoel ik natuurlijk de *entier* van x, met andere woorden, het grootste geheel getal, kleiner dan of gelijk aan x.

Stelling. (*PHP1*) Zij $n \in \mathbb{N}_0$. Als men meer dan n duiven over n duivenhokken verdeelt, dan bestaat er duivenhok dat minstens twee duiven bevat.

Bewijs. Als elk duivenhok ten hoogste één duif zou bevatten, dan zou het totaal aantal duiven (in alle n duivenhokken samen) niet groter zijn dan n. Dit is een contradictie, dus er bestaat een duivenhok dat minstens twee duiven bevat. □

We herformuleren deze stelling: als A en B eindige verzamelingen zodat $|A| > |B|$, en als $f : A \to B$ een afbeelding is, dan kan f niet injectief zijn. Zie je waarom dit precies dezelfde stelling is?

Stelling. (*PHP2*) Als men oneindig veel duiven over een eindig aantal duivenhokken verdeelt, dan bestaat er een duivenhok dat oneindig veel duiven bevat.

Bewijs. Als elk duivenhok een eindig aantal duiven zou bevatten, dan zou het aantal duiven in alle duivenhokken samen eindig zijn (omdat er slechts een eindig aantal duivenhokken is). Dit is een contradictie, dus een van de duivenhokken bevat oneindig veel duiven. □

Als je “professioneel” wil klinken dan kan je de stelling ook als volgt formuleren: als A een oneindige verzameling is, als B een eindige verzameling is en als $f : A \to B$ een afbeelding is, dan bestaat er een oneindige verzameling $C \subset A$ zodat alle elementen van C hetzelfde beeld hebben onder f.

De meest algemene vorm van het duivenhokprincipe (en de meeste krachtige) is de volgende:

Stelling. (*PHP3*) Zijn $n, k \in \mathbb{N}_0$. Als men n duiven verdeelt over k duivenhokken, dan bestaat er een duivenhok dat minstens $\left[\frac{n-1}{k}\right] + 1$ duiven bevat.

(Of nog: als m en n natuurlijk getallen zijn, en men verdeelt meer dan mn duiven over n duivenhokken, dan bestaat er een duivenhok dat minstens $m + 1$ duiven bevat.)

Bewijs. Als elk duivenhok minder dan $\left[\frac{n-1}{k}\right] + 1$ duiven zou bevatten, dan zou het totaal aantal duiven in alle duivenhokken samen niet groter zijn dan $k\left[\frac{n-1}{k}\right]$. Nu is het duidelijk dat $[x] \leq x$ voor alle reële getallen x. Bijgevolg is $k \cdot \left[\frac{n-1}{k}\right] \leq n-1$. Het totaal aantal duiven zou dus niet groter zijn dan $n-1$, maar dat is onmogelijk aangezien er n duiven zijn. Bijgevolg bestaat er een duivenhok dat minstens $\left[\frac{n-1}{k}\right]+1$ duiven bevat. $\square$

De herformulering van de vraag laat ik deze keer aan jou over.

3 Minstens twee Vlamingen hebben evenveel hoofdharen!

In dit hoofdstukje presenteer ik een aantal mooie (en soms moeilijke) toepassingen van het duivenhokprincipe. Het eerste voorbeeldje is echter verre van moeilijk...

Voorbeeld 0. Er bestaan twee Vlamingen die evenveel hoofdharen hebben.
Bewijs. Een mens heeft niet niet meer dan 200.000 hoofdharen (een bekend biologisch feit). Omdat er veel meer dan 200.000 niet-kale Vlamingen zijn, moeten minstens twee Vlamingen evenveel haren op hun hoofd hebben, volgens PHP1. $\square$

Voilà, tijd voor de serieuze voorbeelden...

Voorbeeld 1. (*VWO 1989*) Bewijs dat elke deelverzameling met 55 elementen van de verzameling $S = \{1, 2, 3, ..., 100\}$ twee getallen bevat waarvan het (positieve) verschil gelijk is aan 9.
Oplossing. Zij A een deelverzameling van S met $|A| = 55$ en zijn $a_1 < a_2 < ... < a_{55}$ de elementen van A. Definieer $b_i = a_i + 9$ voor $i = 1, 2, 3, ..., 55$. Dan geldt $b_{55} = a_{55} + 9 \leq 109$. We hebben dus 110 natuurlijke getallen a_i, b_i, allen kleiner dan 110 en verschillend van 0. Volgens PHP1 moeten twee van deze getallen gelijk zijn, zodat $a_i = b_j$ voor zekere indices i en j, en $a_i - a_j = 9$. $\square$
Er bestaan verschillende alternatieve oplossingen voor deze vraag; je kan bijvoorbeeld restklassen modulo 9 beschouwen. Volgens PHP3 moeten minstens 7 elementen van A tot dezelfde restklasse behoren; noem deze getallen $b_1 < b_2 < ... < b_7$. We veronderstellen dat er geen indices i en j bestaan zodat $a_i - a_j = 9$. Omdat de zeven getallen b_i tot dezelfde restklasse behoren (mod 9) zal $b_j - b_i \geq 18$, waarbij $1 \leq i < j \leq 7$. Dan is $b_7 - b_1 = (b_7 - b_6) + (b_6 - b_5) + ... + (b_2 - b_1) \geq 6 \cdot 18 = 108$. Dat is natuurlijk onmogelijk voor twee getallen die tot S behoren; de veronderstelling was dus foutief. $\square$

Soms kan het duivenhokprincipe in een meetkundige context worden toegepast:

Voorbeeld 2. In een vierkant waarvan de zijde lengte 1 heeft liggen 51 punten. Bewijs dat er een cirkelschijf met straal $\frac{1}{7}$ bestaat die minstens 3 van de gegeven punten bedekt.
Oplossing. Verdeel het vierkant in 25 congruente vierkantjes met zijden van lengte $\frac{1}{5}$. Volgens PHP3 bestaan er dan 3 punten die in hetzelfde vierkant liggen. Noem dit vierkant V en noem O het middelpunt van dit vierkant. De omgeschreven cirkel van V heeft middelpunt O en straal $\frac{1}{5\sqrt{2}} < \frac{1}{7}$. De cirkel met middelpunt O en straal $\frac{1}{7}$ zal het vierkant V bijgevolg volledig bedekken, en bijgevolg zullen ook alle punten binnen V (dit zijn er minstens 3) bedekt worden door deze cirkel. $\square$

Het volgende voorbeeldje is een schitterende toepassing van het duivenhokprincipe:

Voorbeeld 3. Bewijs dat elke verzameling van 13 reële getallen twee getallen a en b bevat zodat

$$0 \leq \frac{a-b}{1+ab} \leq 2 - \sqrt{3}.$$

Oplossing. Zij $S = \{s_1, s_2, ..., s_{13}\}$ de gegeven verzameling en stel $t_i = \text{bgtan}\, s_i$ voor $i = 1, 2, ..., 13$. Dan geldt voor $i = 1, 2, ..., 13$ dat $t_i \in [-\pi/2, \pi/2]$. We verdelen het interval $[-\pi/2, \pi/2]$ in twaalf deelintervallen van gelijke lengte, namelijk de intervallen $[k\pi/12, (k+1)\pi/12]$ voor $k = -6, -5, ..., 5$. Omdat er 13 getallen t_i gegeven zijn zullen twee getallen tot hetzelfde interval behoren (PHP1), met andere woorden, er bestaan indices p en q zodat $0 \leq t_p - t_q \leq \pi/12$. Omdat de tangensfunctie stijgend is op het interval $]-\pi/2, \pi/2[$ geldt er dat $0 \leq \tan(t_p - t_q) \leq \tan(\pi/12)$. Merk nu op dat $\tan(\pi/12) = 2 - \sqrt{3}$ en dat

$$\tan(t_p - t_q) = \frac{\tan t_p - \tan t_q}{1 + \tan t_p \cdot \tan t_q} = \frac{s_p - s_q}{1 + s_p s_q}.$$

Stel $s_p = a$ en $s_q = b$: we zijn nu klaar. $\square$

Soms moet je een opgave veralgemenen om die te kunnen oplossen:

Voorbeeld 4. (*Servië-Montenegro 2002*) Toon aan dat er een natuurlijk getal $k \neq 0$ bestaat zodat de cijfers 3, 4, 5 en 6 niet voorkomen in de decimale voorstelling van het getal $k \cdot 2002!$.
Oplossing. We bewijzen een veel mooiere stelling: alle natuurlijke getallen n hebben een veelvoud van de vorm $11\ldots100\ldots0$. (In dit geval is natuurlijk $n = 2002!$) Zij $a_k = 11\ldots11$ het getal dat uit k cijfers 1 bestaat. Natuurlijk bestaan er oneindig veel dergelijke getallen, maar er zijn slechts eindig veel restklassen modulo n. Volgens PHP2 bestaan er dus indices p en q zodat $a_p \equiv a_q \pmod{n}$. Bijgevolg is $n | a_p - a_q$, en $a_p - a_q$ is van de vorm $11\ldots100\ldots0$, dus we zijn klaar. $\square$

De volgende drie voorbeelden zijn zeer moeilijke opgaven. Je zal zien dat de toepassing van het duivenhokprincipe vaak slechts een klein stukje is van de volledige oplossing van een opgave.

Voorbeeld 5. Gegeven is een rechthoekig rooster van punten met 13 rijen en 13 kolommen. Men kleurt 53 van de 169 gegeven punten rood. Bewijs dat er een rechthoek bestaat waarvan de zijden evenwijdig zijn aan de randen van het rooster en waarvan alle hoekpunten rode roosterpunten zijn.
Oplossing. Noem a_1, a_2, ..., a_{13} het aantal rode punten in de eerste, tweede, ..., dertiende rij respectievelijk. Natuurlijk is $a_1 + a_2 + ... + a_{13} = 53$. Na enig nadenken zien we dan dat er een dergelijke rechthoek bestaat als

$$\sum_{k=1}^{13} \binom{a_k}{2} > \binom{13}{2} = 78. \ (*)$$

Inderdaad, voor rij i zijn er $\binom{a_i}{2}$ mogelijke koppels van rode punten in die rij. Beschouw nu de 13 kolommen van het rooster. Er zijn $\binom{13}{2} = 78$ mogelijke combinaties van twee kolommen. Als nu $(*)$ geldt, dan zal een zekere combinatie van twee kolommen minstens twee maal bereikt worden door twee paren van punten (PHP1), waarbij de twee punten van elk paar in dezelfde rij liggen en waarbij de twee paren onderling in verschillende rijen gelegen zijn. (Dit klinkt moeilijk maar dat is het niet.) Het volstaat dus om te bewijzen dat $(*)$ geldt. Welnu,

$$\sum_{k=1}^{13} \binom{a_k}{2} = \sum_{k=1}^{13} \frac{a_k(a_k-1)}{2} = \frac{1}{2} \cdot \sum_{k=1}^{13} a_k^2 - \frac{1}{2} \cdot \sum_{k=1}^{13} a_k.$$

Nu is $a_1 + a_2 + ... + a_{13} = 53$ dus geldt volgens de ongelijkheid van Cauchy dat

$$\left(a_1^2 + a_2^2 + ... + a_{13}^2\right)\left(1^2 + 1^2 + ... + 1^2\right) \geq \left(a_1 + a_2 + ... + a_{13}\right)^2 \Rightarrow \sum_{k=1}^{13} a_k^2 \geq \frac{53^2}{13}.$$

Bijgevolg geldt inderdaad dat

$$\frac{1}{2} \cdot \sum_{k=1}^{13} a_k^2 - \frac{1}{2} \cdot \sum_{k=1}^{13} a_k \geq \frac{1}{2} \cdot \left(\frac{53^2}{13} - 53\right) > 78$$

dus moet er een dergelijke rechthoek bestaan. $\square$

Voorbeeld 6. (*IMO 1987 Vraag 3*) Zijn x_1, x_2, ..., x_n reële getallen zodat $x_1^2 + x_2^2 + ... + x_n^2 = 1$. Bewijs dat, $\forall k \in \mathbb{N}$ zodat $k \geq 2$, er gehele getallen a_1, a_2, ..., a_n bestaan, niet allen gelijk aan 0, zodat $|a_i| \leq k - 1$ voor alle i en zodat

$$|a_1x_1 + a_2x_2 + ... + a_nx_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

Oplossing. Zonder verlies van de algemeenheid mogen we veronderstellen dat $x_1 \geq x_2 \geq ... \geq x_n$. Zij m de unieke index waarvoor geldt dat $x_1, x_2, ..., x_m \geq 0$ en $x_{m+1}, ..., x_n < 0$. (Als alle getallen x_i strikt negatief zijn, dan stellen we $m = 0$; als alle getallen x_i positief zijn dan stellen we $m = n$.) Zij $\mathcal{C}$ de verzameling van alle vectoren $(c_1, c_2, ..., c_n) \in \mathbb{R}^n$ zodat $c_i \in \{0, 1, ..., k-1\}$. Beschouw alle mogelijke waarden van de som $S = c_1x_1 + c_2x_2 + ... + c_nx_n$ voor $(c_1, c_2, ..., c_n) \in \mathcal{C}$. De kleinst en grootst mogelijke waarden van S worden respectievelijk bereikt voor

$$(c_1, c_2, ..., c_n) = (\underbrace{0, ..., 0}_{m}, \underbrace{k-1, ..., k-1}_{n-m});$$

$$(c_1, c_2, ..., c_n) = (\underbrace{k-1, ..., k-1}_{m}, \underbrace{0, ..., 0}_{n-m}).$$

Noem deze extreme waarden A en B respectievelijk, dan geldt er dat

$$B - A = (k-1)(|x_1| + |x_2| + ... + |x_n|).$$

Omdat $x_1^2 + x_2^2 + ... + x_n^2 = 1$ geldt volgens de ongelijkheid van Cauchy dat

$$(x_1^2 + x_2^2 + ... + x_n^2)(1 + 1 + ... + 1) \geq (|x_1| + |x_2| + ... + |x_n|)^2 \Rightarrow |x_1| + |x_2| + ... + |x_n| \leq \sqrt{n}.$$

Bijgevolg is $B - A \leq (k-1)\sqrt{n}$. Nu liggen alle waarden die S kan aannemen in het interval $[A, B]$. We verdelen dit interval in $N = k^n - 1$ deelintervallen van gelijke lengte, namelijk van lengte $(B - A)/N$. Omdat de verzameling $\mathcal{C}$ precies k^n vectoren bevat, bestaan er volgens PHP1 twee vectoren $(c'_1, c'_2, ..., c'_n)$ en $(c''_1, c''_2, ..., c''_n)$ waarvoor de corresponderende sommen S' en S'' in hetzelfde deelinterval liggen, waarbij

$$S' = c'_1x_1 + c'_2x_2 + ... + c'_nx_n, \quad S'' = c''_1x_1 + c''_2x_2 + ... = c''_nx_n.$$

Dan geldt dus $|S' - S''| \leq (B - A)/N$. Neem nu $a_1 = c'_i - c''_i$ voor $i = 1, 2, ..., n$. Op die manier verkrijgen we een vector $(a_1, a_2, ..., a_n)$, verschillend van de nulvector in $\mathbb{R}^n$, waarvoor geldt dat $|a_i| \leq k - 1$ voor $i = 1, 2, ..., n$ en

$$|a_1x_1 + a_2x_2 + ... + a_nx_n| = |S' - S''| \leq \frac{B - A}{N} \leq \frac{(k-1)\sqrt{n}}{k^n - 1}$$

dus we hebben gehele getallen $a_1, a_2, ..., a_n$ gevonden die voldoen aan het te bewijzen. □

Voorbeeld 7. (*IMO 2001 Vraag 3*) Aan een wiskundecompetitie namen 21 jongens en 21 meisjes deel. Achteraf bleek dat geen enkele deelnemer meer dan 6 problemen oploste, en dat er voor elke jongen en voor elk meisje minstens één probleem bestaat dat door zowel die jongen als door dat meisje opgelost werd. Bewijs dat er een probleem bestaat dat opgelost werd door minstens 3 jongens en minstens 3 meisjes.

Oplossing. Om te beginnen voeren we een paar notaties in. Noem J de verzameling van de jongens, M de verzameling van de meisjes, P de verzameling van de problemen. Dan is $|M| = |J| = 21$. Noem $P(m)$ de verzameling van de problemen die opgelost werden door een meisje $m \in M$ en $P(j)$ de verzameling van de problemen die opgelost werden door $j \in J$. Tenslotte noemen we $J(p)$ en $M(p)$ respectievelijk de verzamelingen van de jongens en meisjes die het probleem $p \in P$ oplossen. Nu kunnen we aan de slag. We willen bewijzen dat er een probleem $p \in P$ bestaat waarvoor geldt dat $|J(p)| \geq 3$ en $|M(p)| \geq 3$. We veronderstellen dat dit niet het geval is. We zullen op twee verschillende manieren het aantal elementen tellen van de verzameling $T = \{(p, m, j) \mid p \in P(m) \cap P(j)\}$. Natuurlijk is $|T| \geq 21^2 = 441$, omdat er 21^2 koppels (j, m) bestaan van een jongen en een meisje en omdat er voor elk koppel (j, m) een probleem bestaat dat zowel door j als door m opgelost werd. Veronderstel dus dat er geen $p \in P$ bestaat zodat $|M(p)| \geq 3$ en $|J(p)| \geq 3$. We merken op dat

$$\sum_{m \in M} |P(m)| = \sum_{p \in P} |M(p)| \leq 6\,|M| = 126, \ \sum_{j \in J} |P(j)| = \sum_{p \in P} |J(p)| \leq 6\,|J| = 126$$

omdat geen enkele deelnemer meer dan 6 problemen oplost. (De gelijkheden hierboven kan men eenvoudig aantonen en zijn eigenlijk niet meer dan logisch.) We definiëren nu

$$P_+ = \{p \in P \mid |M(p)| \geq 3\}; \ P_- = \{p \in P \mid |M(p)| \leq 2\}.$$

We zullen bewijzen dat

$$\sum_{p \in P_-} |M(p)| \geq |M|\,, \ \sum_{p \in P_+} |J(p)| \geq |J|\,. \ (*)$$

Zij $m \in M$ een willekeurig meisje. Volgens het duivenhokprincipe lost m een probleem p op dat door minstens $\left[\frac{21-1}{6}\right] + 1 = 4$ jongens opgelost wordt. Wegens onze veronderstelling volgt uit $|J(p)| \geq 4$ dat $p \in P_-$, dus elk meisje lost minstens één probleem uit P_- op. Op analoge wijze toont men aan dat elke jongen minstens één probleem uit P_+ oplost. Daarmee is $(*)$ bewezen. Dan geldt ook

$$\sum_{p \in P_+} |M(p)| = \sum_{p \in P} |M(p)| - \sum_{p \in P_-} |M(p)| \leq 5\,|M|\,, \ \sum_{p \in P_-} |J(p)| \leq 5\,|J|\,.$$

Nu is

$$|T| = \sum_{p \in P} |M(p)| \cdot |J(p)| = \sum_{p \in P_+} |M(p)| \cdot |J(p)| + \sum_{p \in P_-} |M(p)| \cdot |J(p)|$$

en bijgevolg geldt volgens onze veronderstelling dat

$$|T| \leq 2 \sum_{p \in P_+} |M(p)| + 2 \sum_{p \in P_-} |J(p)| \leq 10\,|M| + 10\,|J| = 420.$$

We bewezen echter dat $|T| \geq 441$. Onze veronderstelling was dus foutief; we zijn dus klaar. □

4 Al doende leert men!

De eerste opgaven zijn zeer eenvoudig, de laatste opgaven zijn aan de moeilijke kant (maar natuurlijk niet zo moeilijk als Voorbeeld 7.) De laatste opgave is vraag 6 van IMO 2005: normaalgezien is zo'n vraag onoplosbaar voor een Belg, maar in 2005 was vraag 6 iets gemakkelijker dan gewoonlijk, en maar liefst 2 van de 3 Vlaamse IMO-deelnemers losten deze vraag op! Succes!

1. Bewijs dat er een getal N van de vorm 20042004...2004 bestaat waarvoor geldt:
   - N is deelbaar door 2003, en
   - N heeft niet meer dan 10.000 cijfers in decimale voorstelling.
2. (*IMO 1972 Vraag 1*) Bewijs: elke verzameling van 10 natuurlijke getallen kleiner dan 100, heeft twee disjuncte deelverzamelingen waarvan de sommen van de elementen gelijk zijn.
3. Bewijs: elk veelvlak heeft twee zijvlakken die begrensd worden door een gelijk aantal zijden.
4. (*British Mathematical Olympiad 2000*) Bestaat er een verzameling van elf gehele getallen zodat geen zes van deze gehele getallen een zesvoud als som hebben?
5. Een basketbalteam speelde 45 wedstrijden in een maand met 30 dagen. Elke dag speelde het team minstens één wedstrijd. Bewijs dat er een periode van een bepaald aantal dagen bestaat zodat het team gedurende die periode precies 14 wedstrijden speelde.
6. (*Rusland 1961*) Men plaatst 120 vierkantjes met zijden van lengte 1 binnen een rechthoek met afmetingen 20×25, en dat op willekeurige wijze. Bewijs dat men een cirkel met diameter 1 binnen deze rechthoek kan plaatsen zodat deze cirkel geen enkel vierkantje snijdt.
7. (*Bulgarian Mathematical Olympiad 2003*) Bart en Ria spelen het volgende spel. Bart schrijft n verschillende natuurlijke getallen op een papier, met n een natuurlijk getal. Ria mag enkele van deze getallen wegstrepen (ze mag er ook geen enkel wegstrepen, maar ze mag ze zeker niet allemaal wegstrepen). Daarna mag Ria voor elk van de overblijvende getallen een + of een − zetten en de som bepalen van de getallen die ze op deze manier bekomt. Als deze som deelbaar is door 2003 dan wint Ria. In het andere geval wint Bart. Voor welke waarden van n heeft Bart een winnende strategie, en voor welke waarden van n heeft Ria een winnende strategie?

8. (*IMO 1988 Longlist*) Zijn $a_1, a_2, \ldots, a_{11} \in \mathbb{Z}$. Bewijs dat er $b_1, b_2, \ldots, b_{11} \in \{-1, 0, 1\}$ bestaan (niet allen gelijk aan 0) zodat $\sum_{k=1}^{11} a_k b_k$ deelbaar is door 2005.

9. (*China 1990*) Bepaal het kleinste natuurlijk getal n zodat er voor elke verzameling $\{a_1, a_2, ..., a_n\}$ van n verschillende reële getallen, gekozen uit het interval $[1, 1000]$, indices i en j bestaan waarvoor geldt dat $0 < a_i - a_j < 1 + 3\sqrt[3]{a_i a_j}$.

10. Binnen een cirkel met straal 16 liggen 650 gegeven punten. Definieer een *ring* als het vlakdeel dat begrepen is tussen twee concentrische cirkels met stralen 2 en 3 respectievelijk. Bewijs dat men een ring kan plaatsen zodat minstens 10 van de 650 punten bedekt worden door deze ring.

11. (*IMO 2005 Vraag 6*) Bij een wiskundewedstrijd kregen de deelnemers 6 opgaven voorgelegd. Ieder tweetal van deze opgaven werd door meer dan $\frac{2}{5}$ van het aantal deelnemers opgelost. Niemand loste alle 6 de opgaven op. Laat zien dat minstens 2 deelnemers ieder precies 5 opgaven hebben opgelost.

○ ○ ○

terug naar echt bestand

6.4 meer meetkunde

GEOMETRY
for the
OLYMPIAD ENTHUSIAST

Bruce Merry

The South African Mathematical Society (SAMS) has the responsibility for selecting teams to represent South Africa at the Pan African Mathematics Olympiad (PAMO) and the International Mathematical Olympiad (IMO).

Team selection begins with the Old Mutual Mathematical Talent Search, a self-paced correspondence course in problem-solving that starts afresh in January each year. The best performers in the Talent Search are invited to attend mathematical camps in which they learn specialised problem-solving skills and write challenging Olympiad-level papers. Since the Pan African Maths Olympiad is not quite as daunting as the International version, the tradition has evolved that South African PAMO teams consist of students who have not previously been selected for an IMO team. The Inter-Provincial Mathematical Olympiad and the South African Mathematics Olympiad are closely linked with the PAMO and IMO selection programme.

To provide background reading for the Talent Search, the South African Mathematical Society has published a series of Mathematical Olympiad Training Notes that focus on mathematical topics and problem-solving skills needed in mathematical competitions and Olympiads. Six booklets have appeared to date:

- *The Pigeon-hole Principle*, by Valentine Goranko
- *Topics in Number Theory*, by Valentin Goranko
- *Inequalities for the Olympiad Enthusiast*, by Graeme West
- *Graph Theory for the Olympiad Enthusiast*, by Graeme West
- *Functional Equations for the Olympiad Enthusiast*, by Graeme West
- *Mathematical Induction for the Olympiad Enthusiast*, by David Jacobs

Though their primary target is the development of high-level problem-solving skills, these booklets can be read by anybody interested in the mathematics just beyond the high school curriculum. They are therefore particularly useful to teachers looking for enrichment material, and students who plan to study mathematics at university level and would like more of a challenge than the school curriculum provides.

For more information, write to

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South Africa's participation in the Pan African and International Mathematical Olympiads is supported by Old Mutual and the Department of Science and Technology.

John Webb
January 2003

Geometry for the Olympiad Enthusiast

Bruce Merry

A booklet in this series was last published in 1996, and the series has been somewhat dormant since. Geometry has long been a gap in this series, and eventually I decided to address this gap. I started writing this booklet in December 2000. It was then put aside for three years, while I focused on my studies. In December 2003 I finally returned to finish the rather delayed project.

This booklet is primarily about classical, or Euclidean, geometry. Trigonometry is used as a tool, but is not explored in great depth, and coordinate geometry barely puts in an appearance. While tackling the exercises and geometry problems in general, one should remember that trigonometry and coordinate geometry are powerful tools. I simply did not have much to say about them.

The booklet assumes a knowledge of high-school geometry. If you have not completed the high-school syllabus, it would be a good idea to first find a textbook and work through both the theory and the exercises. The proofs included here are somewhat terse and you may need to fill in a few details yourself.

Some important results are left as problems, so you should at the very least read the problems (although you really should attempt to solve them, as well). The positioning of problems in the book is a good indicator of how you are expected to tackle them, although of course there are usually other solutions. There are two types of problems: exercises that deal very specifically with the topic in hand, and real olympiad problems. The olympiad problems are labelled with a star ($\star$). The exercises are generally easier than the olympiad problems, but some of them are quite challenging.

I would like to thank Dirk Laurie for writing his Geomplex diagram drawing package. This book would not have been possible without it. I would also like to thank Mark Berman, whose flair for geometry has always inspired me to find elegant solutions.

Contents

1 Techniques

Geometry is unlike many of the other areas of olympiad mathematics, requiring more intuition and less algebra. Nevertheless, it is important to do the basic groundwork as otherwise your intuition has nothing with which to work.

Here are some suggestions on ways to approach a geometry problem.

- Draw a quick diagram so that you can visualise the problem.
- Draw a neat and accurate diagram — this will often reveal additional facts which you could then try to prove.
- Draw a deliberately incorrect diagram (this could be your initial diagram), so that you don't accidentally assume the result because you referred to your accurate diagram (this is particularly important if you are proving concurrency or collinearity).
- It is very important to do as much investigation as you can. Try to relate as many angles and line segments as you can, even if you have several variables. Then look for similar or congruent triangles, parallel lines and so on. This on its own can be enough to solve some easier problems without even having to think.
- There are many approaches to attack geometry problems e.g. Euclidean geometry, coordinate geometry, complex numbers, vectors and trigonometry. Think about applying all the ones that you know to the problem and deciding which ones are most likely to work. Be guided by what you are asked to prove: for example, if you are asked to prove that two lines are parallel then coordinate geometry might work well, but if the problem involves lots of related angles then trigonometry may be a better approach.
- Don't be afraid to get your hands dirty with trigonometry, coordinate geometry or algebra. While such solutions might not be as "cool" as solutions that require an inspired construction, they are often easier to find and score the same number of points. However, doing as much as possible with Euclidean geometry first can make the equations simpler.
- Look for constructions that will give you similar triangles, special angles or allow you to restate the problem in a simpler way. For example, if you are asked to prove something about the sum of two lengths, try making a construction that places the two lengths end to end so that you only have to prove something about the length of a single line.

- Assume that the result is true, and see what follows from this. This may lead you to intermediate results which you can then try to prove.
- Always check that you haven't omitted any cases such as obtuse angles or constructions that are impossible in certain cases (for example, you can't take the intersection point of two lines if they are parallel). This booklet does a terrible job of this, because the special cases are almost always trivial. I'm lazy, the duplication costs of this booklet are high, the rainforests are dying, and this is not a competition. In a competition, you can expect to lose marks if your proof does not work in all cases.

2 Terminology and notation

There is some basic terminology for things that share some property. Concurrent lines pass through a common point, and collinear points lie on a common line. Concyclic points lie on a common circle; note that "A, B, C and D are concyclic" does not have the same meaning as "$ABCD$ is a cyclic quadrilateral", since the latter implies that the points lie in a particular order around the circle. Concentric circles have a common centre.

The humble triangle has possibly the richest terminology and notation. There are numerous "centres", generally the point of concurrency of certain lines, and a few have corresponding circles.

incentre The centre of the incircle (inscribed circle); the point of concurrency of the internal angle bisectors

circumcentre The centre of the circumcircle (circumscribed circle); the point of concurrency of the perpendicular bisectors

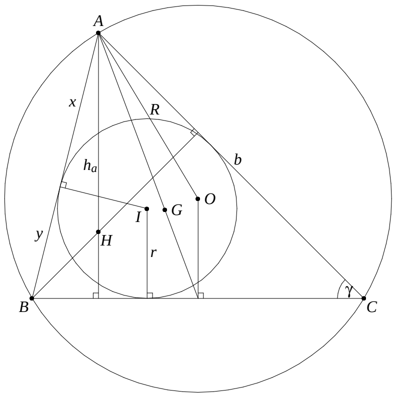
excircle The centre of an excircle (escribed circle); the point of concurrency of two external and one internal angle bisector

orthocentre The point of concurrency of the altitudes

centroid The point of concurrency of the medians (lines from a vertex to the midpoint of the opposite side)

Most of these terms should be familiar from high-school geometry. An unfamiliar term is a *cevian*: this is any line joining a vertex to the opposite side.

For this booklet (particularly section 6), we also introduce a lot of notation for triangles. Some of this is standard or mostly standard while some is not; you are advised to define any of these quantities in proofs, particularly K, x, y and z.



I the incentre

I_A the excentre opposite A

O the circumcentre

G the centroid

H the orthocentre

a the side opposite vertex A (similarly for B and C)

s the semiperimeter, $\frac{a+b+c}{2}$

x the tangent from A to the incircle, $\frac{-a+b+c}{2} = s - a$ (similarly for y and z)

R the radius of the circumcircle (circumradius)

r the radius of the incircle (inradius)

r_a the radius of the excircle opposite A

h_a the height of the altitude from A to BC

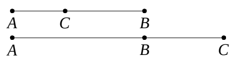
α the angle at A (similarly for β and γ)

K the area of the triangle

We also use the notation $|\triangle ABC|$ (or just $|ABC|$) to indicate the area of $\triangle ABC$.

3 Directed angles, line segments and area

In classical geometry, most quantities are undirected. That means that if you measure them in the opposite direction, they have the same value ($AB = BA$, $\angle ABC = \angle CBA$, and $|\triangle ABC| = |\triangle CBA|$). Most of the time this is a reasonable way of doing things. However, it occasionally has disadvantages. For example, if you know that A, B and C are collinear, and $AB = 5$, $BC = 3$, then what is AC? It could be either 2 or 8, depending on which way round they are on the line. The same problem arises when adding angles or areas.



Normally these situations are not important, because it is clear from a diagram which is correct. However, sometimes there are many different ways to draw the diagram, leading to a proof with many different cases. Another way to solve the problem is to treat the quantities as having a sign, indicating the direction. So now if you are told that $AB = 5$, $BC = 3$ then you can be sure that $AC = AB + BC = 8$. This is because both have the same sign, and hence are in the same direction. If C lay between A and B, then $AB = 5$, $BC = -3$ and so $AC = AB + BC = 2$. It could also be that $AB = -5$, $BC = 3$; the positive direction is generally arbitrary but must be consistent. What is important is that no matter in what order A, B and C lie, the equation $AC = AB + BC$ holds.

Directed line segments have somewhat limited use, because it only makes sense to compare lines that are parallel. Generally they are used when dealing with ratios or products of collinear line segments (see Menelaus' Theorem (6.3), for example). Directed angles and directed area are more often used.

A directed angle $\angle ABC$ is really a measure of the angle between the two lines AB and BC. Conventionally, it is the amount by which AB must be rotated anti-clockwise to line up with BC. One effect of this is that while normal angles have a range of 360°, directed angles only have a range of 180°! This is because rotating a line by 180° leaves it back where it started, so 180° is equivalent to 0°. To indicate this, equivalent angles are sometimes written $\angle ABC \equiv \angle DEF$ rather than $\angle ABC = \angle DEF$. This limitation occasionally has disadvantages, and in particular it is not generally possible to combine trigonometry with directed angles (since the sin and cos functions only repeat every 360°). This is made up for by the special properties that directed angles do have:

1. $\angle AMC \equiv \angle AMB + \angle BMC$;
2. $\angle AXY \equiv \angle AXZ$ iff X, Y, Z are collinear
3. $\angle XYZ \equiv 0°$ iff X, Y, Z are collinear
4. $\angle ABC + \angle BCA + \angle CAB \equiv 0$;
5. $\angle PQS \equiv \angle PRS$ iff P, Q, R and S are concyclic.

Property 1 is simply the basis of directedness: the relative positions don't matter. Property 2 is trivial if Y and Z lie on the same side of X, and the fact that adjacent angles add up to 180° if not. Property 3 just restates the fact that rotating a line onto itself leads to no rotation. Property 4 is the result that angles in a triangle add up to 180°, but also brings in the fact that the three angles are either all clockwise or all anti-clockwise. Property 5 is the really interesting one: it is *simultaneously* the same segment theorem and the alternate segment theorem, depending on the ordering of the points on the circle. The problem below illustrates why having a single theorem can be so important.

Directed areas are used even less often than directed angles and line segments, but are sometimes useful when adding areas to compute the area of a more complex shape. Conventionally, a triangle ABC has positive area if A, B and C are arranged in anti-clockwise order, and negative if they are arranged in clockwise order.

Exercise 3.1. *Three circles, Γ_1, Γ_2 and Γ_3 intersect at a common point O. Γ_1 and Γ_2 intersect again at X, Γ_2 and Γ_3 intersect again at Y, and Γ_3 and Γ_1 intersect against at Z. A is a point on Γ_1 which does not lie on Γ_2 or Γ_3. AX intersects Γ_2 again at B, and BY intersects Γ_3 again at C. Prove that A, Z and C are collinear.*

Exercise 3.2 (Simpson Line). *Perpendiculars are dropped from a point P to the sides of $\triangle ABC$ to meet BC, CA, AB at D, E, F respectively. Show that D, E and F are collinear if and only if P lies on the circumcircle of $\triangle ABC$.*

You will find that directed angles in particular play a large role in the theorems in this book, and they are introduced early on for this purpose. Do not be led to believe that directed angles are so wonderful that they should be used for all problems: theorems try to make very general statements and use directed angles for generality, but most problems are constrained so that normal angles are adequate (e.g. points inside triangles or acute angles). Normal angles are easier to work with simply because one does not need to think about whether to write $\angle ABC$ or $\angle CBA$.

4 Trigonometry

Trigonometry is seldom required to solve a problem. After all, trigonometry is really just a way of reasoning about similar triangles. However, it is a very powerful reasoning tool, and if applied correctly can replace a page full of unlikely and ungainly constructions with a few lines of algebra. If applied incorrectly, however, it can have the opposite effect.

The first thing to do before applying any trigonometry is to reduce the number of variables to the minimum. Then choose the variables that you want to keep very carefully. The compound angle formulae below make it easy to expand out many trig expressions, but if you have chosen the wrong variables to start with the task is almost impossible.

The following angle formulae are invaluable in manipulating trigonometric expressions. In the formulae below, a $\mp$ indicates a sign that is opposite to the sign chosen in a $\pm$.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{4.1}$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{4.2}$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \tag{4.3}$$
$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot A \pm \cot B} \tag{4.4}$$
$$\sin A \sin B = [\cos(A - B) - \cos(A + B)]/2 \tag{4.5}$$
$$\sin A \cos B = [\sin(A - B) + \sin(A + B)]/2 \tag{4.6}$$
$$\cos A \cos B = [\cos(A - B) + \cos(A + B)]/2 \tag{4.7}$$
$$\sin A \pm \sin B = 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right) \tag{4.8}$$
$$\cos A + \cos B = 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \tag{4.9}$$

$$\cos A - \cos B = 2\sin\left(\frac{B+A}{2}\right)\sin\left(\frac{B-A}{2}\right) \tag{4.10}$$

You don't need to memorise any of these other than the first three, because all the others can be obtained from these with simple substitutions. You should be aware that these transformations exist and know how to derive them, so that you can do so in an olympiad if necessary (see the exercises).

You can also use these to derive other formulae; for example, you can calculate $\sin n\theta$ and $\cos n\theta$ in terms of $\sin\theta$ and $\cos\theta$ fairly easily (for small, known values of n).

Exercise 4.1. *Prove equations* (4.4) *to* (4.10)*.*

Exercise 4.2. *In a $\triangle ABC$ (which is not right-angled), prove that*

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

4.1 The extended sine rule

The standard Sine Rule says that

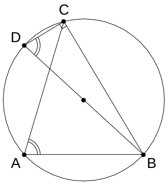
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Theorem 4.1 (Extended Sine Rule). *In a triangle ABC,*

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R is the radius of the circumcircle.

Proof. Construct point D diametrically opposite B in the circumcircle of $\triangle ABC$. Then $\alpha = \angle CDB$ or $180° - \angle CDB$ and $\angle BCD = 90°$. It follows that $\frac{a}{\sin\alpha} = \frac{BC}{BC/BD} = 2R$, and similarly for $\frac{b}{\sin\beta}$ and $\frac{c}{\sin\gamma}$.



□

Exercise 4.3. *In a circle with centre O, AB and CD are diameters. From a point P on the circumference, perpendiculars PQ and PR are dropped onto AB and CD respectively. Prove that the length of QR is independent of the position of P.*

5 Circles

5.1 Cyclic quadrilaterals

A cyclic quadrilateral is a quadrilateral that can be inscribed in a circle. There are several results related to the angles of a cyclic quadrilateral that are covered in high school mathematics and which will not be repeated here. These results are still very important, and cyclic quadrilaterals appear in many unexpected places in olympiad problems.

Exercise 5.1 ($\star$). *Let $\triangle ABC$ have orthocentre H and let P be a point on its circumcircle. Let E be the foot of the altitude BH, let PAQB and PARC be parallelograms, and let AQ meet HR in X.*

(a) Show that H is the orthocentre of $\triangle AQR$.

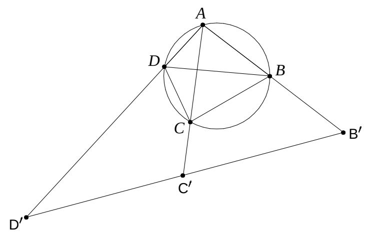
(b) Hence, or otherwise, show that EX is parallel to AP.

A result that is not normally taught in school is Ptolemy's Theorem. It is mainly useful if you have only one or two cyclic quadrilaterals, and lengths play a major role in the problem. It is also very useful when some more is known about the lengths. Equal lengths are particularly helpful as they can divide out of the equation.

Theorem 5.1 (Ptolemy's Theorem). *If ABCD is a cyclic quadrilateral, then*

$$AB\cdot CD + BC\cdot AD = AC\cdot BD$$

Proof.



Choose an arbitrary constant K and construct B', C' and D' on AB, AC and AD respectively such that $AB\cdot AB' = AC\cdot AC' = AD\cdot AD' = K$.

Now consider $\triangle ABC$ and $\triangle AC'B'$. The angle at A is common and $\frac{AB}{AC'} = \frac{K/AB'}{K/AC} = \frac{AC}{AB'}$ and therefore the triangles are similar. It follows similarly that $\triangle ABD|||\triangle AD'B'$ and $\triangle ACD|||\triangle AD'C'$. Hence $\angle B'C'D' = \angle ABC + \angle ADC = 180°$ i.e. $B'C'D'$ is a straight line. From the similar triangles, we have $BC = B'C'\cdot\frac{AB}{AC'} = \frac{B'C'\cdot AB\cdot AC}{K}$, and similarly for CD and BD. Therefore

$$\begin{aligned}AC\cdot BD &= \tfrac{B'D'}{K}(AB\cdot AC\cdot AD)\\ &= (\tfrac{B'C'}{K}+\tfrac{C'D'}{K})(AB\cdot AC\cdot AD)\\ &= AB\cdot CD + AD\cdot BC\end{aligned}$$

This result relies on the fact that $B'C'D'$ is a straight line. If we had used a non-cyclic quadrilateral, this would not have been the case. This shows that the converse of Ptolemy's Theorem is also true. In fact the triangle inequality in $\triangle B'C'D'$ leads to *Ptolemy's Inequality*, which says that $AC\cdot BD \leq AB\cdot CD + AD\cdot BC$ for any quadrilateral $ABCD$, with equality precisely for cyclic quadrilaterals. □

Exercise 5.2. *Triangle ABC is equilateral. For any point P, show that $AP+BP \geq CP$ and determine when equality occurs.*

5.2 The Simpson line

The Simpson line was covered as exercise 3.2, but to emphasise its importance the statement is repeated here. A handy corollary is that the feet of perpendiculars from a point on the circumcircle cannot all meet the sides internally — which can limit the number of cases you need to consider.

Theorem 5.2 (The Simpson line). *Perpendiculars are dropped from a point P to the sides of $\triangle ABC$ to meet BC, CA, AB at D, E, F respectively. Show that D, E and F are collinear if and only if P lies on the circumcircle of $\triangle ABC$.*

This was exercise 3.2, so no proof is provided here.

Exercise 5.3. *From a point E on a median AD of $\triangle ABC$ the perpendicular EF is dropped to BC, and a point P is chosen on EF. Then perpendiculars PM and PN are drawn to the sides AB and AC.*

Now, it is most unlikely that M, E and N will lie in a straight line, but in the event that they do, prove that AP bisects $\angle A$.

5.3 Power of a point

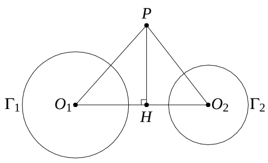
This section is based on the fact that if chords AB and CD of a circle intersect at a point P, then $PA\cdot PB = PC\cdot PD$ (even if P lies outside the circle). This is easily shown using similar triangles.

Consider fixing a point P and circle Γ and considering all possible chords AB that pass through P. Since $PA\cdot PB$ is equal for every pair of chords AB, it is equal for *all* such chords. This value is said to be the power of P with respect to Γ. The line segments are considered to be directed (see section 3), so P is negative inside the circle and positive outside of it. In fact by considering the chord that passes through O, the centre of Γ, it can be seen that the power of P is $d^2 - r^2$, where $d = OP$ and r is the radius of Γ. If P lies outside the circle then this also equals the square of the length of the tangent from P to Γ.

It is sometimes useful to know that the converse of the above result is true i.e. if $PA\cdot PB = PC\cdot PD$, where AB and CD pass through P, then A, B, C and D are concyclic (but only if using directed line segments).

5.3.1 The radical axis

Consider having two circles instead of one. What is the set of points which have the same power with respect to both circles? If the circles are concentric then no point will have the same power (since d will be the same and r different for every point), but the situation is less clear in general.



Consider two circles Γ_1 and Γ_2 with centres O_1 and O_2 with radii r_1 and r_2 respectively. Let P be a point which has equal powers with respect to Γ_1 and Γ_2, and let H be the foot of the perpendicular from P onto O_1O_2. Then

$$
\begin{aligned}
& O_1P^2 - r_1^2 = O_2P^2 - r_2^2 && (5.1)\\
\iff\quad & O_1H^2 + HP^2 - r_1^2 = O_2H^2 + HP^2 - r_2^2 && (5.2)\\
\iff\quad & O_1H^2 - r_1^2 = O_2H^2 - r_2^2 && (5.3)\\
\iff\quad & O_1H^2 - r_1^2 = (O_2O_1 - HO_1)^2 - r_2^2 && (5.4)\\
\iff\quad & 2 \cdot HO_1 \cdot O_2O_1 = O_2{O_1}^2 + r_1^2 - r_2^2 && (5.5)
\end{aligned}
$$

We have eliminated P from the equation! In fact (5.3) shows that P has equal powers with respect to the circles iff H does. If $O_1O_2 \neq 0$ then we have a linear equation in HO_1 and so there is exactly one possibility for H (we are using directed line segments, so HO_1 uniquely determines H). Thus the locus of P is the line through H perpendicular to O_1O_2. This line is known as the *radical axis* of Γ_1 and Γ_2.

If the two circles intersect, the radical axis is easy to construct. The points of intersection both have zero power with respect to both circles, so both points lie on the radical axis. So the radical axis is simply the line through them.

Exercise 5.4. *Two circles are given. They do not intersect and neither lies inside the other. Show that the midpoints of the four common tangents are collinear.*

5.3.2 Radical centre

What happens when we consider three circles (say Γ_1, Γ_2 and Γ_3) instead of two? Firstly consider the case where the centres are not collinear. Then the radical axis of Γ_1 and Γ_2 will meet the radical axis of Γ_2 and Γ_3 at some point, say X (they will not be parallel because a radical axis is perpendicular to the line between the centres of the circles). Then from the definition of a radical axis, X has the same power with respect to all three circles and so it also lies on the radical axis of Γ_1 and Γ_2. The fact

that the three radical axes are concurrent at a point (known as the *radical centre*) can be used to solve concurrency problems.

If, however, the three centres are collinear, then all three radical axes are parallel. If they all coincide then all points on the common axis have equals powers with respect to the three circles; if not then no points do.

Exercise 5.5. *Show how to construct, using ruler and compass, the radical axis of two non-intersecting circles.*

Exercise 5.6 (⋆). *Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.*

6 Triangles

6.1 Introduction

A triangle would seem to be almost the simplest possible object in geometry, second only to the circle. It has only has two true degrees of freedom, since scaling a triangle up or down does not affect its properties. Yet the humble triangle contains an enormous amount of mathematics — in fact too much to fully explore here.

6.2 Tangents to the incircle

Let the lengths of the tangents to the incircle from A, B and C be x, y and z. Since $a = y + z$, $b = z + x$ and $c = x + y$, we can solve for x, y and z and get

$$x = \frac{-a+b+c}{2}, \quad y = \frac{a-b+c}{2}, \quad z = \frac{a+b-c}{2}.$$

This is the same notation that is introducted in section 2.

Exercise 6.1. *Determine the lengths of the tangents from B and C to the excircle opposite A.*

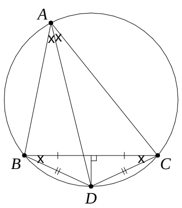
6.3 Triangles within triangles

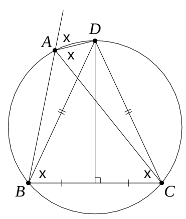
There are specific names given to certain triangles formed from points of the original triangle:

- The *medial* triangle has the midpoints of the original sides as its vertices.
- The *orthic* triangle has the feet of the altitudes as its vertices.
- A *pedal* triangle is the triangle formed by the feet of perpendiculars dropped from some point onto the three sides. If the point is the orthocentre, then this is the orthic triangle (and in fact some people use the term "pedal triangle" to refer to the orthic triangle).

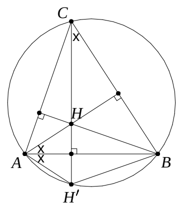
6.4 Points on the circumcircle

Apart from the vertices, there are a few other points that are known to lie on the circumcircle. The first is the intersection point of a perpendicular bisector and the corresponding angle bisector. This is easily shown by taking the intersection of the perpendicular bisector and the circumcircle, which divides an arc (say BC) into two equal parts which subtend equal angles at A. This is also true (although less well known) in the case where the *external* angle bisector is used.





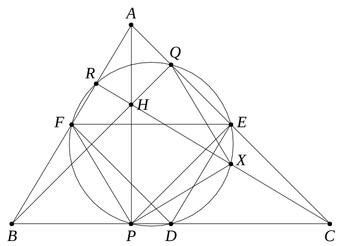
The second group of points that are known to lie on the circumcircle are the reflections of H (the orthocentre) in each of the three sides. This is an exercise in angle chasing, using the known results about the angles in cyclic quadrilaterals.



Exercise 6.2. *A rectangle HOMF has $HO = 23$ and $OM = 7$. Triangle ABC has orthocentre H and circumcentre O. The midpoint of BC is M and F is the foot of the altitude from A. Determine the length of side BC.*

6.5 The nine-point circle

A rather interesting circle that arises in a triangle is the so-called nine-point circle. Let us examine the circumcircle of the triangle whose vertices are the midpoints of $\triangle ABC$ (the medial triangle). Firstly, what is its radius? The medial triangle is a half sized version of the original triangle (because of the midpoint theorem), so its circumradius will also be half that of the large triangle, i.e. it will be $\frac{R}{2}$.



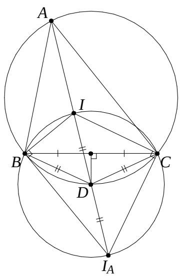
Now let us see what other points this circle passes through. From the diagram it appears that it passes through the feet of the altitudes, so let us prove this. Since F is the midpoint of the hypotenuse of $\triangle APB$, we have $\angle FPA = \angle FAP = 90° - \beta$. Similarly $\angle EPA = 90° - \gamma$ and so $\angle FPE = \alpha = \angle FDE$ (since $\triangle ABC \,|||\, \triangle DEF$). It follows that P lies on the circle. Similarly Q and R also lie on the circle.

Point X is the midpoint of HC, and it also appears to lie on the circle. HC is the diameter of the circle passing through H, Q, C and P, so X is the centre of this circle. It follows that $\angle PXQ = 2\angle PCQ = 2\gamma$. But $\angle PEQ = \angle PEF + \angle FEQ = \angle PDF + \angle FEQ = \gamma + \gamma$, so $\angle PEQ = \angle PXQ$ and so X lies on the circle. Similarly the midpoints of HA and HC lie on the circle.

Because there are nine well-defined points which lie on this circle, it is known as the nine-point circle.

6.6 Another circle

Consider that $\angle I_ABI = \angle I_ACI = 90°$; this shows that II_A is the diameter of a circle passing through I, I_A, B and C. Where is the centre of this circle? Well, any circle passing through B and C must have its centre on the perpendicular bisector of BC, and for II_A to be the diameter, the centre must also lie on the internal bisector of $\angle A$. Hence the centre is the intersection of these two lines. As shown above, the intersection also lies on the circumcircle of $\triangle ABC$.



Exercise 6.3 (⋆). *In acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that*

(i) the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$;

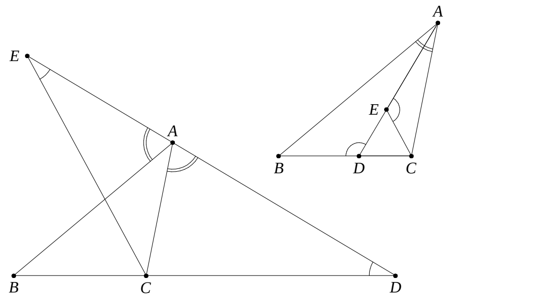
(ii) the area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

6.7 Theorems

Angle bisectors can be fairly tricky to deal with. The angle bisector theorem provides a way to compute the segments which the base is divided into.

Theorem 6.1 (Angle bisector theorem). *If D is the point of intersection of BC with an angle bisector of $\angle A$, then $\frac{DB}{DC} = \frac{AB}{AC}$.*

Proof. Construct E on AD such that $\angle AEC = \angle BDA$. Then $\triangle ABD \,|||\, \triangle ACE$ (two angles) and so $\frac{DB}{EC} = \frac{AB}{AC}$. But $\triangle ECD$ is isosceles, so $CE = CD$ and therefore $\frac{DB}{DC} = \frac{AB}{AC}$ as required.



□

Exercise 6.4. *In the right-hand diagram for the angle-bisector theorem, find a formula for the length BD in terms of the side lengths a, b and c.*

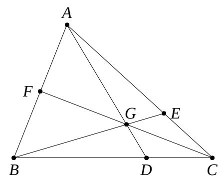
Exercise 6.5. *Given a line segment AB and a real number $r > 0$, find the locus of points P such that $\frac{AP}{BP} = r$.*

The theorems of Ceva and Menelaus are handy results when proving concurrency and collinearity respectively. They are particularly powerful because their converses are true, provided that the directions are taken into account. The converses are quite easy to prove by assuming them to be false, and then constructing two different points with the same uniquely defining properties.

Theorem 6.2 (Ceva's Theorem). *If AD, BE and CF are concurrent cevians of $\triangle ABC$ then*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Proof.



Let G be the point of concurrency.

$$\frac{|\triangle ABD|}{|\triangle ACD|} = \frac{BD}{DC} \quad \text{(common height)}$$
$$\frac{|\triangle GBD|}{|\triangle GCD|} = \frac{BD}{DC} \quad \text{(common height)}$$
$$\therefore \quad \frac{|\triangle AGB|}{|\triangle CGA|} = \frac{BD}{DC}$$

We can show similar things for $\frac{CE}{EA}$ and $\frac{AF}{FC}$. Therefore

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{|\triangle AGB|}{|\triangle CGA|} \cdot \frac{|\triangle BGC|}{|\triangle AGB|} \cdot \frac{|\triangle CGA|}{|\triangle BGC|} = 1$$

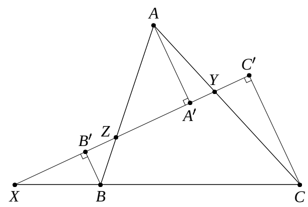
This proof has not explicitly invoked directed areas or line-segments, but if they are used it can be seen that the result will hold even if G lies outside of the triangle. □

Theorem 6.3 (Menelaus' Theorem). *If X, Y and Z and collinear and lie on sides BC, CA and AB (or their extensions) of $\triangle ABC$ respectively, then*

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1$$

(Note that the sign on the result is due to directed line segments, and indicates that the line cuts the sides themselves either twice or not at all.

Proof.



Drop perpendiculars from A, B and C to meet XYZ at A', B' and C'. From alternate angles, we have $\triangle AA'Z \,|||\, \triangle BB'Z$ and thus $\frac{AZ}{ZB} = \frac{AA'}{B'B}$. Similarly $\frac{BX}{XC} = \frac{BB'}{C'C}$ and $\frac{CY}{YA} = \frac{CC'}{A'A}$. Therefore

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{AA'}{B'B} \cdot \frac{BB'}{C'C} \cdot \frac{CC'}{A'A} = -1$$

□

Exercise 6.6. *Use Menelaus' Theorem to prove Ceva's Theorem.*

Exercise 6.7 ($\star$). *ABC is an isosceles triangle with $AB = AC$. Suppose that*

(i) M is the midpoint of BC and O is the point on the line AM such that $OB \perp AB$;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F lies on the line AC such that E, Q and F are distinct and collinear.

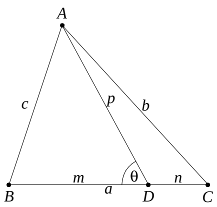
Prove that OQ is perpendicular to EF if and only if $QE = QF$.

Stewart's Theorem is a handy tool for dealing with the length of a cevian, which is otherwise difficult to work with.

Theorem 6.4 (Stewart's Theorem). *Suppose AD is a cevian in $\triangle ABC$. Let $p = AD$, $m = BD$ and $n = CD$. Then*

$$a(p^2 + mn) = b^2 m + c^2 n.$$

Proof.



Use the cosine rule in $\triangle ABD$:

$$c^2 = m^2 + p^2 - 2mp\cos\theta$$
$$\therefore \quad c^2 n = m^2 n + p^2 n - 2mnp\cos\theta \tag{6.1}$$

Do the same in $\triangle ACD$, noting that $\cos(180^\circ - \theta) = -\cos\theta$:

$$b^2 = n^2 + p^2 + 2np\cos\theta$$
$$\therefore \quad b^2 m = n^2 m + p^2 m + 2mnp\cos\theta \tag{6.2}$$

Now add (6.1) and (6.2):

$$b^2 m + c^2 n = m^2 n + n^2 m + p^2 n + p^2 m \tag{6.3}$$
$$= (m+n)(p^2 + mn) \tag{6.4}$$
$$= a(p^2 + mn) \tag{6.5}$$

□

In the special case that AD is a median, Stewart's Theorem reduces to $4p^2 + a^2 = 2(b^2 + c^2)$, which is known as Apollonius' Theorem.

Exercise 6.8. *In $\triangle ABC$, angle A is twice angle B. Prove that $a^2 = b(b+c)$.*

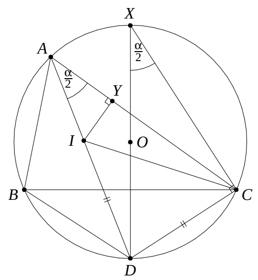
Theorem 6.5 (Euler's Formula).

$$OI^2 = R(R - 2r)$$

As a corollary, we have Euler's Inequality:

$$R \geq 2r.$$

Proof. Extend the angle bisector from A to meet the circumcircle again at D. Also construct X diametrically opposite D on the circumcircle and construct Y as the foot of the perpendicular from I onto AC. We calculate the power of I with respect to the circumcircle (see section 5.3), which is equal to $OI^2 - R^2$ and also to $-AI \cdot ID$. From section 6.6, we have $ID = CD$.



Now we note that $\triangle DXC \,|||\, \triangle IAY$, and so $\frac{AI}{IY} = \frac{XD}{DC} \iff AI \cdot ID = 2rR$. Since $OI^2 - R^2 = -AI \cdot ID$, it follows that $OI^2 = R(R - 2r)$ as required. □

Euler's Theorem provides a measure of the distance between the incentre and circumcentre. However it is most often invoked as Euler's Inequality.

Exercise 6.9 (⋆). *Let r be the inradius and R the circumradius of ABC and let p be the inradius of the orthic triangle of triangle ABC. Prove that*

$$\frac{p}{R} \leq 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2.$$

6.8 Area

There are numerous formulae for the area of a triangle, and in many cases things can be discovered by equating them.

Theorem 6.6 (Heron's Formula).

$$K = \sqrt{sxyz}$$

Proof. This is probably the ugliest proof in this booklet. Here goes:

$$\begin{aligned}
16K^2 &= 4(ab\sin\gamma)^2 \\
&= 4a^2b^2(1-\cos^2\gamma) \\
&= 4a^2b^2\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] \\
&= 4a^2b^2-(a^2+b^2-c^2)^2 \\
&= (2ab-a^2-b^2+c^2)(2ab+a^2+b^2-c^2) \\
&= \left[c^2-(a-b)^2\right]\left[(a+b)^2-c^2\right] \\
&= (c-a+b)(c+a-b)(a+b+c)(a+b-c) \\
&= 16sxyz.
\end{aligned}$$

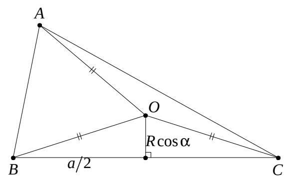
□

Theorem 6.7 (Triangle area formulae).

$$\begin{aligned}
K &= \tfrac{1}{2}ah_a = \tfrac{1}{2}bh_b = \tfrac{1}{2}ch_c && (6.6)\\
&= \tfrac{1}{2}ab\sin\gamma = \tfrac{1}{2}bc\sin\alpha = \tfrac{1}{2}ca\sin\beta && (6.7)\\
&= \frac{abc}{4R} && (6.8)\\
&= 2R^2\sin\alpha\sin\beta\sin\gamma && (6.9)\\
&= \tfrac{1}{2}R(a\cos\alpha+b\cos\beta+c\cos\gamma) && (6.10)\\
&= R(a\cos\beta\cos\gamma+b\cos\gamma\cos\alpha+c\cos\alpha\cos\beta) && (6.11)\\
&= rs && (6.12)\\
&= r_ax = r_by = r_cz && (6.13)\\
&= \sqrt{sxyz} \qquad \textit{(Heron's Formula)} && (6.14)
\end{aligned}$$

Proof. The first is the standard formula for the area of a triangle. The second is really the same formula, since $\sin\gamma = \frac{h_a}{b}$. The third is obtained using the extended sine rule ($\sin\gamma = \frac{c}{2R}$). The fourth is similarly obtained using the extended sine rule by converting all side lengths to sines.

Equation 6.9 is obtained by adding the areas of the isosceles triangles $\triangle BOC$, $\triangle COA$ and $\triangle AOB$. The base of $\triangle BOC$ is a and $\angle BOC = 2\angle BAC = 2\alpha$, so the height is $OC\cos\alpha = R\cos\alpha$. Adding up the areas gives the result.



The following equation is obtained from 6.9 by replacing a by $b\cos\gamma + c\cos\beta$ and similarly for b and c.

Equation 6.12 is obtained similarly to 6.9, but using I instead of O. The three triangles all have height r, so the area is $\frac{1}{2}(ra+rb+rc) = rs$. Equation 6.13 uses the excentre I_a instead; in this case one adds triangles ABI_a and ACI_a and subtracts triangle BCI_a.

Heron's Formula was covered earlier. □

Exercise 6.10. *An equilateral triangle has sides of length* $4\sqrt{3}$. *A point Q is located inside the triangle so that its perpendicular distances from two sides of the triangle are* 1 *and* 2. *What is the perpendicular distance to the third side?*

Exercise 6.11. *Prove that*

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

There is one area more formula that is used with coordinate geometry.

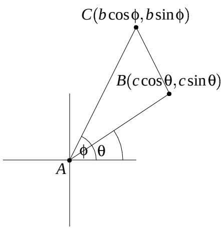
Theorem 6.8. *If one vertex of a triangle is at the origin and the other two are at* (x_1, y_1) *and* (x_2, y_2)*, then*

$$K = \tfrac{1}{2}\left|x_1y_2 - x_2y_1\right|.$$

If the absolute value operator is removed, one gets a formula for directed area[1].

Proof. The proof below uses trigonometry. It is also possible to compute the area of the triangle by starting with a rectangle that bounds it, and subtracting right triangles. However, that approach requires several cases to be considered.

[1]The sign is used in computer graphics to determine whether three points are wound clockwise or anti-clockwise.



Assume without loss of generality that C makes a larger angle from the x-axis than B (swapping B and C simply negates the term inside the absolute value). Then $(x_1, y_1) = (c\cos\theta, c\sin\theta)$, $(x_2, y_2) = (b\cos\phi, b\sin\phi)$ and the area is

$$\begin{aligned}\tfrac{1}{2}bc\sin\alpha &= \tfrac{1}{2}bc\sin(\phi-\theta)\\ &= \tfrac{1}{2}bc(\sin\phi\cos\theta - \cos\theta\sin\phi\\ &= \tfrac{1}{2}(x_1y_2 - x_2y_1).\end{aligned}$$

□

6.9 Inequalities

Inequalities in triangles are often best solved by first expressing all the quantities in terms of as few variables as possible (ideally, only two or three) and then using inequality techniques discussed in *Inequalities for the Olympiad Enthusiast* to finish the problem algebraically. Jensen's Inequality is particularly powerful when combined with trigonometric functions.

Theorem 6.9 (Jensen's Inequality). *A function f is said to be* convex *on an interval $[a,b]$ if $\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$ for all $x, y \in [a,b]$. If f is convex*[2] *on $[a,b]$ then for any $x_1, x_2, \ldots, x_n$ in $[a,b]$ we have*

$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \leq \frac{f(x_1)+\cdots+f(x_n)}{n}.$$

The statement also holds if all inequality signs are reversed, in which case the function is termed concave.

[2] If you are familiar with calculus, a convex function is one that satisfies $f''(x) \geq 0$ for all $x \in [a,b]$.

Proof. Refer to page 18 of *Inequalities for the Olympiad Enthusiast*, by Graeme West. □

Exercise 6.12. *If α, β, γ are the angles of a triangle, then show that $\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\sqrt{3}}{2}$.*

One thing to keep in mind is the triangle inequality: if you reduce the problem to an inequality in a, b and c then it is possible (although not necessarily the case) that you will need to use the fact that the sum of any two is greater than the third. A technique that sometimes simplifies this to substituting $a = x+y$, $b = y+z$, $c = z+x$ in which case the triangle inequality is equivalent to $x, y, z > 0$. In some circles this has become known as the Ravi Substitution, after a Canadian IMO contestant (and later coach) Ravi Vakil. Although he did not invent the technique, he successfully applied it to an IMO problem.

There are a few other useful inequalities that are specific to triangles. The first is Euler's Inequality, mentioned above. The others are listed below.

Theorem 6.10. *In a triangle ABC,*

$$\frac{3\sqrt{3}}{2}R \geq s \qquad s^2 \geq 3\sqrt{3}K \qquad K \geq 3\sqrt{3}r^2.$$

In each case, equality occurs iff $\triangle ABC$ is equilateral.

Proof. We first prove that $\frac{3\sqrt{3}}{2}R \geq s$. From the extended sine rule, $\frac{a}{2R} = \sin\alpha$ and so

$$\begin{aligned}\frac{s}{R} &= \sin\alpha + \sin\beta + \sin\gamma\\ &\leq 3\sin\left(\frac{\alpha+\beta+\gamma}{3}\right) \qquad \text{(Jensen's Inequality)}\\ &= 3\sin 60^\circ\\ &= \frac{3\sqrt{3}}{2}.\end{aligned}$$

For the remaining inequalities, we express everything in terms of x, y and z. Thus

$$\begin{aligned}s^2 &= s^{3/2}\sqrt{s}\\ &= \sqrt{s(x+y+z)^3}\\ &\geq \sqrt{27sxyz} \qquad \text{(AM-GM)}\\ &= 3\sqrt{3}K \qquad \text{(Heron's Formula).}\end{aligned}$$

$$
\begin{aligned}
K &= \frac{r^2 s^2}{K} \\
&\geq \frac{3\sqrt{3}r^2 K}{K} \qquad \text{(from the previous step)} \\
&= 3\sqrt{3}r^2.
\end{aligned}
$$

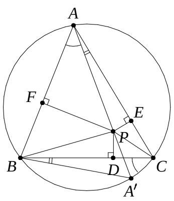
□

Theorem 6.11 (Erdős-Mordell). *Let P be a point inside triangle $\triangle ABC$, and let the feet of the perpendiculars from P to BC, CA, AB be D, E, F respectively. Then*

$$AP + BP + CP \geq 2(DP + EP + FP).$$

Proof. Extend AP to meet the circumcircle of $\triangle ABC$ at A'. Let $\angle BAP = \theta$ and $\angle CAP = \phi$. Note that $FP = AP\sin\theta$ and $EP = AP\sin\phi$, so $\frac{EP}{FP} = \frac{\sin\phi}{\sin\theta} = \frac{CA'}{BA'}$. Also note that $a \cdot AA' = b \cdot BA' + c \cdot CA'$ (from Ptolemy's Theorem in the cyclic quadrilateral $ACA'B$), so $AA' = \frac{b}{a} \cdot BA' + \frac{c}{a} \cdot CA'$. Now

$$
\begin{aligned}
AP &= \frac{FP}{\sin\theta} \\
&= \frac{FP \cdot 2R}{BA'} \qquad \text{(Extended Sine Rule)} \\
&\geq \frac{FP \cdot AA'}{BA'} \qquad (AA' \text{ is less than the diameter}) \\
&= \frac{FP(b \cdot BA' + c \cdot CA')}{a \cdot BA'} \\
&= \frac{b}{a} \cdot FP + \frac{c}{a} \cdot \frac{CA'}{BA'} \cdot FP \\
&= \frac{b}{a} \cdot FP + \frac{c}{a} \cdot EP.
\end{aligned}
$$



Now we can establish similar inequalities for BP and CP, and adding these gives

$$
\begin{aligned}
PA + PB + PC &\geq \left(\frac{b}{c} + \frac{c}{b}\right) PD + \left(\frac{c}{a} + \frac{a}{c}\right) PE + \left(\frac{a}{b} + \frac{b}{a}\right) PF \\
&\geq 2(PD + PE + PF). \qquad \text{(AM-GM)}
\end{aligned}
$$

□

Exercise 6.13. *Let ABC be a triangle and P be an interior point in ABC. Show that at least one of the angles PAB, PBC, PCA is less than or equal to 30 degrees.*

7 Transformations

A very powerful idea in geometry is that of a transformation. A transformation maps every point in space to some other point in space. Structures like lines or circles are transformed by applying the transformation to every point on them. They do not necessary maintain their shapes; in fact there is a transformation (inversion) which generally maps lines to circles! Each transformation will preserve certain properties of a diagram, and by translating the properties of the original into the transformed diagram one can obtain new information. Here a diagram is really just a set of points.

7.1 Affine transformations

The transformations we discuss here are all *affine*. That means that straight lines are mapped to straight lines, and lengths are scaled uniformly. The transformations presented here all preserve angles as well. These transformations can in fact be built up by combining reflections and scale changes, although this is not necessarily the best way to think about them.

7.2 Translations, rotations and reflections

The simplest transformation is a translation: every point simply moves a constant distance in a constant direction; this is like picking up a piece of paper and moving it, without rotating it. Rotations rotate all the points by some angle around a particular point; this is like sticking a pin in a piece of paper and then turning it. Reflections take all points and reflect them in a particular line; this is like picking up the piece of paper and putting it down upside-down (the paper would of course need to be thin enough for the diagram to be seen through the back).

While these are all quite straightforward, they can also be very powerful because they preserve so much. They are also closely related, as shown by the next problem.

Exercise 7.1. *In each of the following, show that the transformations exist using a concrete construction.*

(a) Show that any rotation or translation can be expressed as the combination of a pair of reflections, or vice versa.

(b) Show that two rotations, two translations or a translation and rotation can always be combined to produce a single translation or rotation.

(c) Show that any combination of translations, reflections and rotations yields either a rotation, a translation, or a translation followed by a reflection.

Exercise 7.2. *In acute-angled triangle ABC, a point P is given on side BC. Show how to find Q on CA and R on AB such that $\triangle PQR$ has the minimum perimeter.*

Exercise 7.3 ($\star$). *The point O is situated inside the parallelogram ABCD so that $\angle AOB + \angle COD = 180^\circ$. Prove that $\angle OBC = \angle ODC$.*

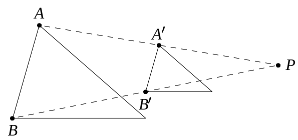
7.3 Homothetisms

So far we have discussed only *rigid* transforms, namely those that can be illustrated with a piece of paper. We now move on to scaling. Imagine drawing a diagram on a new T-shirt, and then letting the T-shirt shrink in the wash. Assume the ink doesn't run and that the T-shirt doesn't warp, you will have the same diagram, only smaller. All the angles and so on will be the same, although lengths will not.

A *homothetism* is a fancy name for scaling. One chooses a centre (sometimes called the "centre of similitude") and a scale factor. Every point is then kept in the same direction relative to the centre, but its distance from the centre is scaled by the scale factor. Like translations, homothetisms preserve orientation, angles, and ratios of lengths. However, lengths are scaled by the scale factor. The result below allows one to find the centre of a homothetism.

Theorem 7.1. *Let S and T be two similar figures which have the same orientation, but are not the same size. Then there is a homothetism that maps S to T.*

Proof. Pick a point A in S and its corresponding point A' in T. Now pick a second point B in S, not on AA', and its corresponding point in B[3]. Now if AA' and BB' are parallel then $AA'B'B$ would be a parallelogram, making $AB = A'B'$. But we assumed that S and T are of different sizes, which would give a contradiction. Hence AA' and BB' meet at a point, which we will call P. Now consider the homothetism with centre of similitude P and scale factor $\frac{A'P}{AP}$. It will clearly map A to A'; will it map B to B'? Yes, because $\triangle ABP \,|||\, \triangle A'B'P$ by parallel lines. If we can show that this homothetism maps the rest of S to T then we are done.



Let C be some arbitrary point in S. We aim to show that the homothetism maps C to its corresponding point C' in T. If C is A or B then we are done. If C lies on AB then C is uniquely defined by $\frac{AC}{BC}$ (with directed line segments). But homothetisms preserve ratios of lengths, and $\frac{A'C'}{B'C'} = \frac{AC}{BC}$ so C is mapped to C'. If C does not lie on AB then C is uniquely defined by the directed angles $\angle BAC$ and $\angle ABC$, and angles are preserved by homothetisms. □

The construction also suggests how the centre of similitude can be found in practice: take two pairs of corresponding points and find the intersection of the lines between them. For example, any two circles of different sizes satisfy the requirements, so a homothetism can be found between them. The points of tangency of the common tangent are corresponding points, since they have the same orientation relative to the centre. Hence the centre of similitude is the intersection of the common tangents.

What happens if we have non-overlapping circles, and use the *other* pair of common tangents? It turns out that this point is also a centre of similitude. However, this homothetism has a negative scale factor, which means that points are "sucked" through the centre and pushed out the other side. This also rotates the figure by 180°, although for a circle this isn't visible. The theorem above in fact applies to situations where the two figures have orientations that are out by 180°, in which case a negative scale factor is used. In this case the figures may even by the same size (since the scale factor is -1, not 1).

Exercise 7.4. *Let ABC be a triangle. Use a homothetism to show that*

(a) the medians of $\triangle ABC$ are concurrent;

[3] If no such B exists, then make some arbitrary construction in S and the corresponding construction in T to produce such a B.

(b) the point of concurrency (the centroid) divides the medians in a $2:1$ *ratio;*

(c) the orthocentre H*, the centroid* G *and the circumcentre* O *are collinear, with* $HG:GO = 2:1$ *(this line is known as the Euler line). Assume that H and O exist (i.e. that the defining lines are concurrent).*

Exercise 7.5 (⋆). *On a plane let C be a circle, L be a line tangent to the circle C and M be a point on L. Find the locus of all points P with the following property: there exist two points* Q,R *on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.*

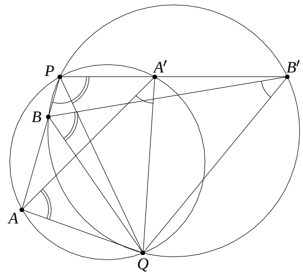
7.4 Spiral similarities

An even more general transformation than a homothetism is a spiral similarity. A spiral similarity combines the effects of a homothetism and a rotation: the plane is not only scaled around a centre P by some factor r, it is also rotated around P by an angle θ. A spiral similarity preserves pretty much the same things as homothetisms i.e. ratio of lengths and angles. However, corresponding lines are no longer parallel, but meet each other at an angle of θ. As for homothetisms, there is a result that makes it possible to find a spiral similarity given two similar figures.

Theorem 7.2. *Let* S *and* T *be two sets of points that are similar but have either different orientation or different size (or both). Then there is a spiral similarity that maps* S *to* T*.*

Proof. In the special case that S and T have the same orientation, there exists a homothetism, which is just a special case of a spiral similarity. So we assume that S and T have different orientations. We also include the case where S and T are oriented 180° apart in the special case, as this is a homothetism with negative scale factor.

Choose two arbitrary points A and B in S, and their corresponding points A' and B' in T. Let P be the intersection of AB and $A'B'$. Construct the circumcircles of $\triangle AA'P$ and $\triangle BB'P$, and let their second point of intersection be Q (Q exists because of the assumptions).



Now $\angle AQA' \equiv \angle APA' \equiv BPB' \equiv BQB'$, $\angle AA'Q \equiv \angle APQ \equiv \angle BPQ \equiv \angle BB'Q$ and similarly $\angle A'AQ \equiv B'BQ$. It follows that triangles $AA'Q$ and $BB'Q$ are directly similar[4]. Now consider the spiral similarity with centre Q, angle AQA' and scale factor $\frac{A'Q}{AQ}$. It will map A to A' by construction, and from the similar triangles it will map B to B'. We can now proceed to show that S is mapped to T, as was done in the corresponding theorem for homothetisms. □

Exercise 7.6. *Squares are constructed outwards on the sides of triangle ABC. Let P, Q and R be the centres of the squares opposite A, B and C respectively. Prove that AP and QR are equal and perpendicular.*

8 Miscellaneous problems

These problems all draw on the techniques in this book, but do not fit well into any particular section. They are mostly very challenging problems designed to give you practice.

Exercise 8.1 (⋆). *ABCD is a square. P is a point inside the square with* $\angle ABP = \angle BAP = 15^\circ$*. Show that* $\triangle CDP$ *is equilateral.*

Exercise 8.2 (⋆). *A 6m tall statue stands on a pedestal, so that the foot of the statue is 2m above your head height. Determine how far from the statue you should stand so that it appears as large as possible in your vision.*[5]

[4] Two triangles are directly similar if they are similar and have the same clockwise/anti-clockwise orientation.

[5] In other words, maximise the angle formed by the foot of the statue, your head and the top of the statue.

Exercise 8.3 ($\star$). *In an acute angled triangle ABC the interior bisector of $\angle A$ intersects BC at L and the circumcircle of $\triangle ABC$ again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas.*

Exercise 8.4 ($\star$). *ABC is a triangle. The internal bisector of the angle A meets the circumcircle again at P. Q and R are similarly defined relative to B and C. Prove that*

$$AP + BQ + CR > AB + BC + CA.$$

Exercise 8.5 ($\star$). *A circle of radius r is inscribed in a triangle ABC with area K. The points of tangency with BC, CA and AC are X, Y and Z respectively. AX intersects the circle again in X'. Prove that $BC \cdot AX \cdot XX' = 4rK$.*

Exercise 8.6 ($\star$). *A semicircle is drawn on one side of a straight line ℓ. C and D are points on the semicircle. The tangents at C and D meet ℓ again at B and A respectively, with the centre of the semicircle between them. Let E be the point of intersection of AC and BD, and F the point on ℓ such that EF is perpendicular to ℓ. Prove that EF bisects $\angle CFD$.*

Exercise 8.7 ($\star$). *In $\triangle ABC$, let D and E be points on the side BC such that $\angle BAD = \angle CAE$. If M and N are, respectively, the points of tangency with BC of the incircles of $\triangle ABD$ and $\triangle ACE$, show that $\frac{1}{MB} + \frac{1}{MD} = \frac{1}{NC} + \frac{1}{NE}$.*

Exercise 8.8 ($\star$). *Let P be a point inside $\triangle ABC$ such that*

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incentres of $\triangle APB, \triangle APC$ respectively. Show that AP, BD and CE meet at a point.

9 Solutions

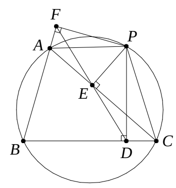
3.1 Using classical geometry to solve this problem would result in an enormous number of different cases. However, directed angles hide all of that, and the result appears with a few lines of basic calculation:

$$\begin{aligned}
\angle AZC &\equiv \angle AZO + \angle OZC \\
&\equiv \angle AXO + \angle OYC && \text{(concyclic points)} \\
&\equiv \angle BXO + \angle OYB && \text{(collinear points)}
\end{aligned}$$

$$\begin{aligned}
&\equiv \angle BXO + \angle OXB && \text{(concyclic points)} \\
&\equiv \angle BXO - \angle BXO && \text{(directed angles)} \\
&\equiv 0^\circ,
\end{aligned}$$

and hence A, Z and C are collinear.

3.2 Note that PC subtends right angles at D and E, and hence is the diameter of a circle passing through P, C, D and E. Similarly, P,A,F and E are concyclic.



$$\begin{aligned}
\angle DEF &\equiv \angle DEP + \angle PEF \\
&\equiv \angle DCP + \angle PAF \\
&\equiv \angle BCP - \angle BAP.
\end{aligned}$$

It follows that $\angle DEF \equiv 0^\circ \iff \angle BCP \equiv \angle BAP$. The first is a condition for D,E,F to be collinear and the second is a condition for P to lie on the circumcircle of $\triangle ABC$.

4.1 $\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot A \pm \cot B}$ can be shown by substituting $\tan\theta = \frac{1}{\cos\theta}$ into $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ and simplifying. The expressions for $\sin A \sin B$ and similar expressions can be proved simply by expanding the right hand side and cancelling terms. The final three equations are derived by making suitable substitutions into the previous three.

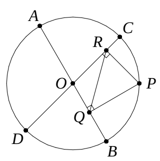
4.2 We first derive a general formula for $\tan(A+B+C)$.

$$\begin{aligned}
\tan(A+B+C) &= \tan[(A+B)+C] \\
&= \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C} \\
&= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B} \cdot \tan C}
\end{aligned}$$

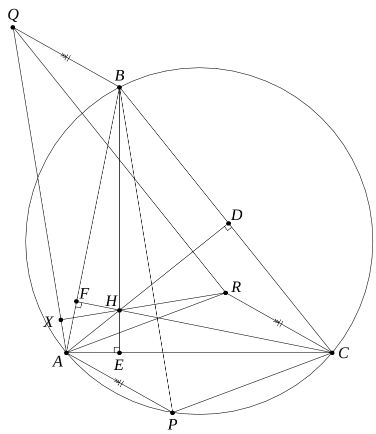
$$= \frac{\tan A + \tan B + (1 - \tan A \tan B)\tan C}{(1 - \tan A \tan B) - (\tan A + \tan B)\tan C}$$
$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

However, we know that $\tan(A+B+C) = \tan 180^\circ = 0$, so the numerator must be 0. The result follows.

4.3 Suppose that P lies on the arc BC, as in the diagram. Then $OQPR$ is cyclic with diameter OP, so applying the extended sine rule in $\triangle OQR$ gives $QR = OP \sin\angle BOC$. Now $\angle BOC$ is fixed and OP is the radius of the circle, also fixed. So QR is fixed if P lies on the arc BC. But $\sin\angle BOC = \sin\angle COA = \sin\angle DOA = \sin\angle DOB$, so QR is constant wherever on the circle P may be.



5.1 This problem is fairly straight-forward as it consists almost entirely of angle chasing. The only difficulty is that P can lie anywhere on the circumcircle, which could give rise to multiple cases. We can get around this with directed angles. This diagram is thus only for reference. D and F are the feet of the altitudes from A and C in $\triangle ABC$.



(a) Firstly notice that since $PAQB$ and $PARC$ are parallelograms, BQ and CR are parallel and equal (and in the same direction), so $BCRQ$ is also a parallelogram. It follows that $RQ \parallel CB$ and hence $AH \perp RQ$. This shows that H lies on one altitude of $\triangle AQR$. If $RX \perp AQ$ then it would lie on another altitude we would be done.

Note that B, D, H and F are concyclic. Thus

$$\begin{aligned}
\angle AHC &\equiv \angle DHF && \text{(opposite angles)}\\
&\equiv \angle DBF && (D, H, F, B \text{ concyclic})\\
&\equiv \angle CBA &&\\
&\equiv \angle CPA && (A, B, C, P \text{ concyclic})\\
&\equiv \angle ARC && (AP \parallel RC, AR \parallel PC)
\end{aligned}$$

and therefore A, H, R and C are concyclic. Thus

$$\begin{aligned}
\angle AXR &\equiv \angle XAR + \angle ARX\\
&\equiv \angle XAB + \angle BAC + \angle CAR + \angle ARH\\
&\equiv \angle QAB + \angle FAC + \angle ACP + \angle ACH
\end{aligned}$$

$$\equiv \angle PBA + \angle ABP + \angle FAC + \angle ACF$$
$$\equiv \angle AFC$$
$$\equiv 90^\circ$$

and the result follows.

(b) This is just more angle chasing, using the fact that H, X, A and E are concyclic (because of the right angles).

$$\begin{aligned}\angle AEX &\equiv \angle AHX \\ &\equiv \angle AHR \\ &\equiv \angle ACR \\ &\equiv \angle PAC \\ &\equiv \angle PAE\end{aligned}$$

from which it follows that $XE \parallel AP$.

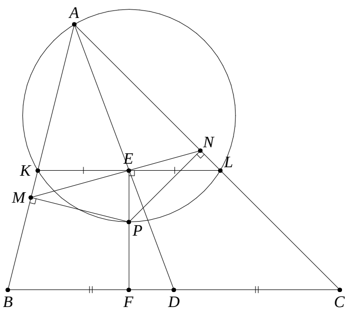
(Proposed for IMO 1996)

5.2 We use Ptolemy's Inequality:

$$\begin{aligned} & AP \cdot BC + BP \cdot CA \geq CP \cdot AB \\ \iff \quad & AP + BP \geq CP \qquad \text{(since } AP = BP = CP\text{)}. \end{aligned}$$

Equality occurs if and only if $ABPC$ is a cyclic quadrilateral.

5.3 Construct KL through E parallel to BC, with K and L on AB and AC respectively.



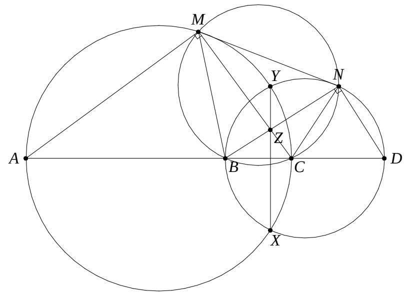
From similar triangles AKE and ABD, we have $KE = BD \cdot \frac{AE}{AD}$. Similarly, $EL = DC \cdot \frac{AE}{AD}$. But $BD = DC$, so $KE = EL$ and hence AE is a median of $\triangle AKL$. Also, $PE \perp KL$ (since $KL \parallel BC$), so M, E and N are the pedal points of P in triangle AKL. The Simpson Line theorem states that M, E and N are collinear if and only if P lies on the circumcircle of $\triangle AKL$. But the perpendicular bisector of KL and the angle bisector of $\angle A$ both meet the circumcircle at the middle of the arc KL, so P lies on the angle bisector of $\angle A$.

(*Crux Mathematicorum*, 1990, 293)

5.4 If P is one of the midpoints, then the lengths of the tangents from P to the two circles are equal. Since these lengths are the square roots of the power of P with respect to these two circles, P must lie on the radical axis. Since this is true for four midpoints, they are collinear because the radical axis is a straight line.

5.5 Call the given circles Γ_1 and Γ_2, and construct a third circle Γ_3 which intersects both Γ_1 and Γ_2. The position of Γ_3 is arbitrary, provided that the centres of the three circles are not collinear. The radical axes of (Γ_1, Γ_2) and (Γ_1, Γ_3) can be found by drawing lines through the intersection points. The intersection of these two lines is the radical centre of the three circles. The desired radical axis now passes through the radical centre and is perpendicular to the line of centres of Γ_1 and Γ_2, which can easily be constructed.

5.6 We use directed angles and line segments, since P may lie either inside or outside of the segment XY. It is also possible (but more tedious) to do the proof with two cases. The diagram below shows the one case.



Label the circle with diameter AC as Γ_1, and the circle with diameter BD as Γ_2. The point Z lies on the radical axis of the two circles, so it has equal power with respect to both. In particular, $ZM \cdot ZC = ZN \cdot ZB$, which prove that M, N, B and C are concyclic. Call this circle Γ_3. Now

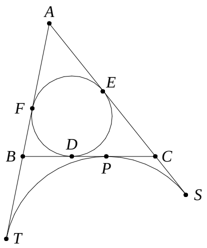
$$\begin{aligned}\angle MND &\equiv \angle MNB + \angle BND \\ &\equiv \angle MCB + 90^\circ \\ &\equiv \angle MCA + \angle AMC \\ &\equiv -\angle CAM \\ &\equiv \angle MAD.\end{aligned}$$

This proves that M, N, A and D are also concyclic; call this circle Γ_4. Finally, we note that AM, DN and XY are the three radical axes formed between the circles Γ_1, Γ_2 and Γ_4. These lines are not all parallel ($AM \parallel XY$ would require that $P = Z$), so they must coincide at the radical centre of the circles.

(IMO 1995, problem 1)

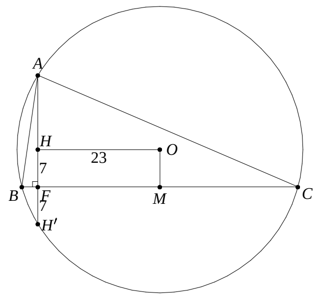
6.1 Let D, E and F be the points of tangency of the incircle with BC, CA, AB and let the excircle be tangent to the same sides at P, S and T respectively. Then from common tangents,

$$\begin{aligned}2ES = 2FT &= ES + FT \\ &= EC + CS + FB + BT \\ &= DC + CP + DB + BP \\ &= 2BC.\end{aligned}$$



Hence $ES = FT = BC = y + z$. Now $BP = BT = FT - BF = (y+z) - y = z$. Similarly, $CP = y$.

6.2 Since the altitude AF passes through H and $BC \perp AF$, it follows that BC and FM coincide. Let H' be the reflection of H in BC. H' is known to lie on the circumcircle of $\triangle ABC$, so $R = H'O = \sqrt{23^2 + 14^2}$. Hence $BM = \sqrt{H'O^2 - 7^2} = 26$ and $BC = 2BM = 52$.



6.3 Clearly, A_0, B_0 and C_0 are in fact I_A, I_B and I_C, and we will refer to them as such.

(i) We will show that $|\triangle II_AC| = 2|\triangle IA_1C|$ (refer to the diagram on page 15, where D is A_1). Results for five other pairs of triangles follow similarly, and adding them all up gives the desired result. Triangles II_AC and IA_1C have a common height, and bases II_A and IA_1. But these bases are the radius and diameter of the circle with diameter II_A, so the result follows.

(ii) It suffices to show that $|AC_1BA_1CB_1|$ is at least twice $|ABC|$, which is equivalent to showing that $|\triangle BCA_1| + |\triangle CAB_1| + |\triangle ABC_1| \geq |\triangle ABC|$. Let A_2, B_2 and C_2 be the reflections of H in BC, CA and AB. These points are known to lie on the circumcircle. When comparing the areas of triangles BCA_1 and BCA_2, we note that they share a common base but the height of $\triangle BCA_1$ is greater than or equal to that of $\triangle BCA_2$. Hence

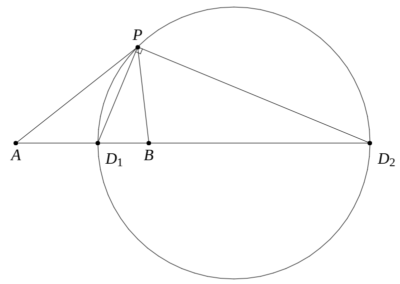
$$\begin{aligned}|\triangle BCA_1| + |\triangle CAB_1| + |\triangle ABC_1| &\geq |\triangle BCA_2| + |\triangle CAB_2| + |\triangle ABC_2| \\ &= |\triangle BCH| + |\triangle CAH| + |\triangle ABH| \\ &= |\triangle ABC|.\end{aligned}$$

(IMO 1989 Question 2)

6.4 Let $BD = m$ and $DC = n$. Then $m+n = a$ and $\frac{n}{m} = \frac{a-m}{m} = \frac{b}{c}$. Hence

$$BD = m = \frac{a}{1+\frac{b}{c}} = \frac{ac}{b+c}.$$

6.5 If $r = 1$, then $AP = BP$ and so the locus is simply the perpendicular bisector. Otherwise suppose $r > 1$ (the situation is symmetric if $r < 1$). Pick an arbitrary P not on AB which satisfies the condition. Let the internal and external angle bisectors of $\angle APB$ meet AB at D_1 and D_2 respectively. Then by the angle bisector theorem, $\frac{AD_1}{BD_1} = \frac{AD_2}{BD_2} = r$. D_1 and D_2 are the only two points on AB that satisfy this, so they are fixed independent of P. Also, $\angle D_1PD_2 = 90°$, so P must lie on the circle with diameter D_1D_2.

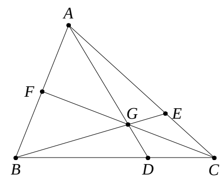


Conversely, suppose P lies on this circle. If P also lies on AB then $P = D_1$ or $P = D_2$, both of which satisfy the conditions. Otherwise let the internal and external bisectors of $\angle PAB$ meet AB at E_1 and E_2 respectively. If $\frac{AP}{BP} = \frac{AE_1}{BE_1} = \frac{AE_2}{BE_2} < r$ then E_1 lies closer to A than D_1 and E_2 lies further from A than D_2. But this means that $\angle E_1PE_2 > 90°$, which is a contradiction. Similarly, if $\frac{AP}{BP} > r$ then $\angle E_1PE_2 < 90°$, again a contradiction. Thus $\frac{AP}{BP} = r$, and this circle is precisely the locus of P.

This circle is known as an *Apollonius* circle.

6.6 Apply Menelaus to $\triangle ACD$ cut by line BGE:

$$\frac{AG}{GD} \cdot \frac{DB}{BC} \cdot \frac{CE}{EA} = -1. \tag{9.1}$$

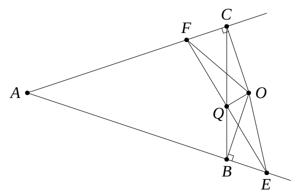


Similarly, one can apply it to $\triangle ABD$ cut by BGE:

$$\frac{AG}{GD} \cdot \frac{DC}{CB} \cdot \frac{BF}{FA} = -1. \tag{9.2}$$

Finally, dividing (9.1) by (9.2) and doing some re-arranging (while being careful with the sign conventions) gives Ceva's Theorem.

6.7 Without loss of generality, let $BQ \leq CQ$, giving the diagram below:



Suppose $OQ \perp EF$. Then $EBQO$ and $FCOQ$ are cyclic quadrilaterals, so $\angle BEO = 180° - \angle BQO = \angle CQO = \angle CFO$. But $BO = CO$, so $\triangle BEO \equiv \triangle CFO$. This gives $EO = FO$, making $\triangle EOF$ isosceles. But $OQ \perp EF$, so $EQ = QF$.

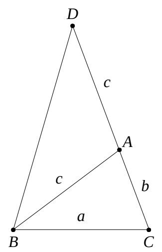
Now suppose that $QE = QF$. Apply Menelaus to triangle AEF, cut by line BQC:

$$1 = \frac{EQ}{QF} \cdot \frac{FC}{CA} \cdot \frac{AB}{BE} = \frac{FC}{BE}.$$

Hence $CF = BE$. Also, $BO = CO$, so $\triangle BEO \equiv \triangle CFO$ and hence $EO = FO$. Then $\triangle EOF$ is isosceles with $EQ = QF$, so $OQ \perp EF$.

(IMO 1994 question 2)

6.8 Construct D on the extension of AC such that $\angle ABD = \angle ABC$. Note that AB is then an angle bisector of $\triangle BDC$. Also, $\angle BDA = 2\angle ABC - \angle ABD = \angle ABD$, so triangle ABD is isosceles and $AD = c$. From the angle bisector theorem (or from $\triangle ABC \,|||\, \triangle BDC$), we find that $AD = \frac{ac}{b}$.



From Stewart's Theorem, we get

$$\begin{aligned} & (b+c)(c^2+bc) = \left(\frac{ac}{b}\right)^2 \cdot b + a^2c \\ \Longrightarrow \quad & (b+c)^2bc = a^2c^2 + a^2bc \\ \Longrightarrow \quad & b(b+c) = a^2, \end{aligned}$$

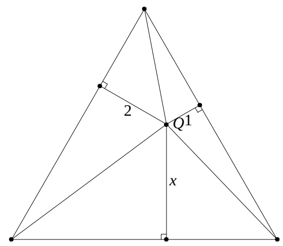
as required.

6.9 Let the orthic triangle be $A'B'C'$. We use Euler's Inequality twice, once on $\triangle ABC$ and once on $\triangle A'B'C'$. The vertices of the orthic triangle lie on the nine-point circle, so the circumradius of $\triangle A'B'C'$ is $R/2$. Thus

$$\begin{aligned} \frac{p}{R} &= \frac{1}{2} \cdot \frac{p}{R/2} \\ &\le \frac{1}{4} \\ &= 1 - \frac{1}{3} \cdot \frac{3^2}{2} \\ &\le 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2. \end{aligned}$$

(Proposed at IMO 1993)

6.10 The height of the triangle is 6, so the area is $12\sqrt{3}$. Let the required length be x, and consider the area as the sum of the areas of the triangles formed by Q and the vertices.



The total area is thus $2\sqrt{3}(1+2+x)$. Solving the equation $12\sqrt{3} = 2\sqrt{3}(1+2+x)$ gives $x = 3$.

6.11 We know that $s = x+y+z$. Divide through by K, recalling that $K = rs = r_a x = r_b y = r_c z$.

6.12 We first check that sin is concave on $[0°, 180°]$:

$$\frac{\sin x + \sin y}{2} = \sin\left(\frac{x+y}{2}\right) \cdot \cos\left(\frac{x-y}{2}\right) \le \sin\left(\frac{x+y}{2}\right).$$

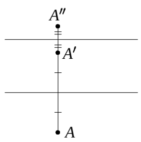
Thus

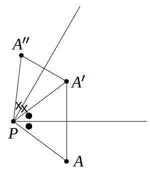
$$\sin\alpha + \sin\beta + \sin\gamma \le 3\sin\left(\frac{\alpha+\beta+\gamma}{3}\right) = 3\sin 60° = \frac{3\sqrt{3}}{2}.$$

6.13 Suppose for a contradiction that these angles are all strictly greater than 30°. Drop perpendiculars from P onto BC, CA, AB to meet at D, E, F respectively. Then $2PF > PA$, $2PD > PB$ and $2PE > PC$. But then $PA + PB + PC < 2(PD + PE + PF)$, which contradicts the Erdős-Mordell Theorem.

(IMO 1991, question 5)

7.1 (a) When combining two reflections, there are two cases.





In the diagrams above, the first reflection maps A to A', and the second maps A' to A''.

(i) The lines of reflection are parallel, separated by a distance d. As can be seen from the diagram, the combination of the reflections is a translation by $2d$, perpendicular to the lines of reflection (the direction depends on the order in which the reflections are performed. Conversely, any translation can be expressed as the combination of two parallel reflections, suitably oriented, and with separation equal to half the distance of the translation.

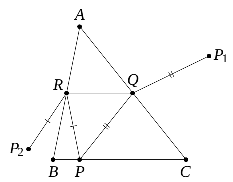
(ii) The lines of reflection are not parallel, and intersect at some point P with an angle of θ. From the diagram, it is now clear that any other point is rotated by an angle of 2θ around P, with the direction depending on the order of the rotations. Conversely, any rotation can be expressed as the combination of two reflections which pass through the centre of the rotation, and with an angle between them of half the rotation angle.

(b) Two translations trivially produce another translation, whose displacement is the vector sum of the original displacements. When one or both of the transformations is a rotation, express the transformations as pairs of reflections. We showed in part (a) that there is some freedom in the choice of reflections. We will have four reflections which are applied in order, say $b_2b_1a_2a_1$.[6] We can always choose the reflections such that a_2 and b_1 are the same. Identical reflections cancel out, so we are left with a_1b_2 which from (a) is equivalent to a rotation or translation.

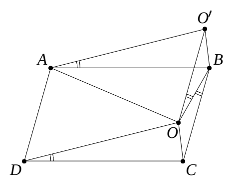
(c) We can transform all the rotations and translations into pairs of reflections, using part (a). We can then pair off these reflections and convert them back into translations and rotations, possibly leaving one reflection at the end. Now part (b) shows that we can reduce the sequence of translations and rotations to just one, which may be followed by a reflection. It remains to show that a rotation followed by a reflection is equivalent to a translation followed by a reflection. We do this by appending two identical (and hence cancelling) reflections to the sequence, at an angle we will choose in a moment. The sequence will now appear as $ccb(r_2r_1)$ where r_2r_1 is the rotation, and c is the newly added reflection. We choose c so that cb forms a rotation with angle exactly opposite to the angle of r_2r_1. Now $(cb)(r_2r_1)$ is the combination of two rotations that forms some translation, say T (it is a translation, not a rotation, because of the choice of angle). Thus the entire sequence is equivalent to cT i.e. a translation followed by a reflection.

[6]We write sequence of transformations from right to left. This is because they are functions, so applying ab to a point P actually means $a(b(P))$, with b being applied first.

7.2 Reflect P in CA to obtain P_1 and reflect P in AB to obtain P_2. Now $PQ+QR+RP = P_1Q+QR+RP_2$. This sum will clearly be smallest when P_1, Q, R and P_2 lie in a straight line. So choose Q and R to be the intersections of P_1P_2 with CA and AB.



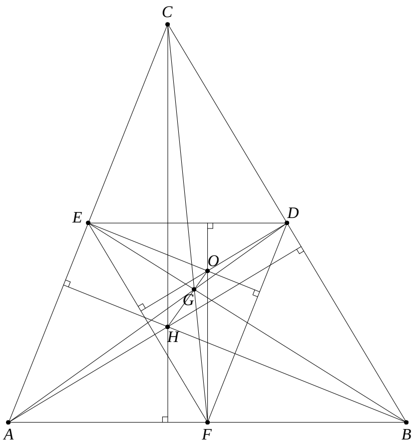
7.3 Having two supplementary angles vertically opposite each other is not very helpful. It would be more useful if we could get the angles to be either adjacent (to create a straight line) or opposite angles of a quadrilateral (to make it cyclic). One way to do this is to "pick up" triangle DOC and place DC on top of AB.



More formally, construct O' outside $ABCD$ such that $\triangle AO'B \equiv \triangle DOC$. Then $\angle AO'B + \angle AOB = 180°$, so $AO'BO$ is cyclic. Also, $OO'BC$ is a parallelogram because $O'B$ and OC are equal and parallel. Thus $\angle OBC = \angle BOO' = \angle BAO' = \angle ODC$.

(Canadian Mathematical Olympiad 1997)

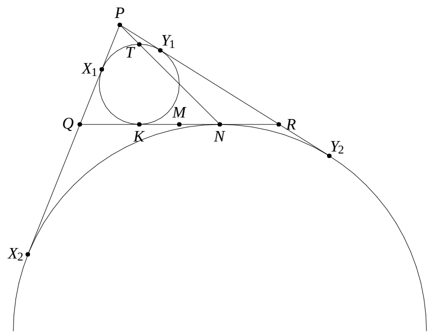
7.4 (a) Let D, E and F be the midpoints of BC, CA and AB respectively. From the Midpoint Theorem, $\triangle DEF \,|||\, \triangle ABC$ and is half the size. It is also oriented $180°$ relative to $\triangle ABC$. Thus there is a homothetism that maps $\triangle ABC$ to $\triangle DEF$, with scale factor $-\frac{1}{2}$. The centre of similitude must lie on AD, BE and CF, and hence these lines are concurrent.



(b) The homothetism maps AG to DG with scale factor $-\frac{1}{2}$, so $AG : GD = 2 : 1$. The result follows similarly for the other two medians.

(c) The line DO is perpendicular to BC, and hence also to EF. Similarly $EO \perp FD$ and $FO \perp DE$, so O is the orthocentre of $\triangle DEF$. Since the homothetism maps $\triangle ABC$ to $\triangle DEF$, it will also map H to O. This proves the collinearity, and the scale follows as in the previous section.

7.5 Start with an arbitrary pair (Q, R) for which P exists, and construct the excircle C_2 of $\triangle PQR$ opposite P (see diagram).



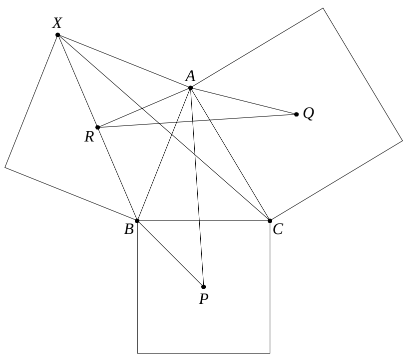
The incircle and excircle of $\triangle PQR$ must be homothetic, and P is the centre of the homothetism. Now let K be the point of tangency of C with L, and let T be the point diametrically opposite K. The corresponding point to T on C_2 must also be vertically above the centre in the diagram, i.e. it is N. But the line through corresponding points must pass through the centre of the homothetism, so P lies on NT.

From the solution to problem 6.1 (page 37), we have $QK = RN$, from which it follows that N and K are symmetrically placed about M. But K and M are fixed, so N must be fixed too.

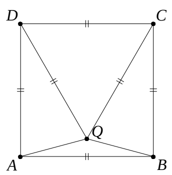
We have now established that any solution P must lie on NT. It is also clear that P must lie strictly beyond T. Conversely, suppose P' is some point on NT beyond T. Let L' be a line through P' and parallel to L, and consider moving a point P along L', finding Q and R on L such that C is the incircle of $\triangle PQR$. When P moves far to the left, the midpoint of QR will be far to the right, and vice versa. Since the midpoint shifts continuously, there is at least one point where it is M. We have shown above that this P must be the intersection of NT with L', namely P', and hence P' satisfies the desired properties. Therefore the locus is the portion of NT that lies strictly beyond T.

(IMO 1992, question 4)

7.6 Consider the spiral similarity with centre A, rotating clockwise (in the diagram) by $45°$ and scaling by $\sqrt{2}$. It will map Q to C and R to X. Now consider the spiral similarity with centre B that rotates anti-clockwise by $45°$ and scales by $\sqrt{2}$. It will map A to X and P to C. These two similarities thus map AP and QR to the same line. They both scale by the same amount ($\sqrt{2}$) and the difference of their angles is $90°$, so AP and QR must be equal and perpendicular.



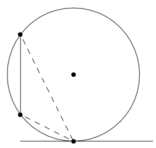
8.1 Construct Q inside the square with $\triangle CDQ$ equilateral. We aim to show that $P = Q$.



Now $\angle QDC = 60°$, so $\angle QDA = 30°$. But $QD = AD$, so $\triangle AQD$ is isosceles and thus $\angle DAQ = 75°$. This makes $\angle BAQ = 15°$, and similarly $\angle ABQ = 15°$. But then triangles ABP and ABQ have two common angles and a common side, so they are congruent. Both P and Q lie on the same side of AB (the inside

of the square), so P and Q must be the same. Triangle CDQ is equilateral by construction, so $\triangle CDP$ is equilateral.

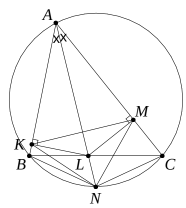
8.2 Construct a circle of radius 5m, with centre 5m above your head height and 4m from the statue. This circle will pass through the head and foot of the statue.



If your head lies on the circle you will have some constant viewing angle θ; with your head inside the circle the angle is larger, and with your head outside the circle it is smaller. But the circle is tangent to the line representing head-height, so the best angle is when your head is at this point of tangency. So you should stand 4m from the statue.

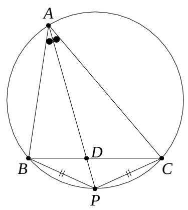
8.3 Firstly note that $\triangle ALK \equiv \triangle ALM$. Hence $AKLM$ is a kite and so $KM \perp AL$; thus $|AKNM| = \frac{1}{2}KM \cdot AN$. Since $ABNC$ is cyclic, $\triangle ABL \,|||\, \triangle ANC$ and hence $AN \cdot AL = AB \cdot AC$. Also, AL is the diameter of the circumcircle of $\triangle AKM$, so $\frac{KM}{AL} = \sin\alpha$. Substituting these into the above gives

$$\begin{aligned} |AKNM| &= \frac{1}{2} \cdot \frac{KM \cdot AB \cdot AC}{AL} \\ &= \frac{1}{2} \cdot AB \cdot AC \cdot \sin\alpha \\ &= |\triangle ABC| \end{aligned}$$



(IMO 1987 Question 2)

8.4 Let D be the point where the angle bisector from A cuts BC.

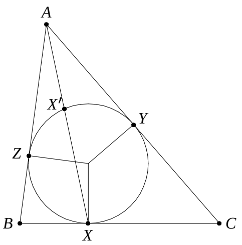


Since $\angle BAD = \angle PAC$ and $\angle DBA = \angle CPA$ we have $\triangle BAD ||| \triangle PAC$. Thus $\frac{c}{BD} = \frac{AP}{PC}$. From exercise 6.4 we have $BD = \frac{ac}{b+c}$. It follows that $AP = \frac{b+c}{a} \cdot PC$. But $PB = PC$ and so from the triangle inequality, $2PC > BC \iff PC > \frac{a}{2}$. Therefore $AP > \frac{b+c}{2}$.

Similarly $BQ > \frac{c+a}{2}$ and $CR > \frac{a+b}{2}$. Adding these inequalities gives the desired result.

(Australian Mathematics Olympiad 1985)

8.5 Firstly note that $AX \cdot AX'$ is the power of A with respect to the incircle, so it is equal to $AZ^2 = x^2$. Thus $a \cdot AX \cdot XX' = a \cdot AX^2 - ax^2$.



We can calculate $a \cdot AX^2$ using Stewart's Theorem:

$$BC(AX^2 + BX \cdot XC) = AC^2 \cdot BX + AB^2 \cdot CX$$

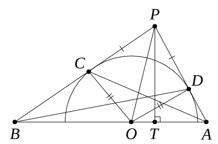
$$\begin{aligned}
a(AX^2 + yz) &= b^2 y + c^2 z \\
a \cdot AX^2 &= (x+z)^2 y + (x+y)^2 z - (y+z)yz \\
&= x^2 y + 2xyz + z^2 y + x^2 z + 2xyz + y^2 z - y^2 z - z^2 y \\
&= x^2(y+z) + 4xyz \\
&= ax^2 + 4xyz.
\end{aligned}$$

Now we can calculate $a \cdot AX^2 - ax^2$

$$\begin{aligned}
a \cdot AX^2 - ax^2 &= 4xyz \\
&= \frac{4}{s} \cdot sxyz \\
&= \frac{4}{s} \cdot K^2 \\
&= \frac{4}{s} \cdot rsK \\
&= 4rK \text{ as desired.}
\end{aligned}$$

(Arbelos May 1987)

8.6 This is a good example of a problem that becomes much easier with a good diagram (the diagram below is intentionally skewed). If AD and BC are extended to meet at P, then it appears that P, E and F are collinear. This would be a useful thing to know, so we attempt to prove it.



Let T be the foot of the perpendicular from P to AB and let O be the centre of the semicircle. $\triangle OCB ||| \triangle PTB$, so $\frac{CB}{TB} = \frac{BO}{BP}$. Similarly $\frac{DA}{TA} = \frac{AO}{AP}$. We want to prove that PT, AC and BD are concurrent, which by the converse of Ceva's Theorem would be true if

$$\frac{PC}{CB} \cdot \frac{BT}{TA} \cdot \frac{AD}{DP} = 1$$

Firstly, $PC = PD$ (equal tangents to the semicircle), and we can substitute the ratios found above to change this to $\frac{BP}{BO} \cdot \frac{AQ}{AP} = 1$. However, this is true by the angle bisector theorem (PD is an angle bisector because $\triangle PCO \equiv \triangle PDO$). It follows that E lies on the altitude from A, and $F = T$.

Now notice that PO subtends right angles at C, D and F, so $PCFD$ is a cyclic quad. Thus $\angle DFP = \angle DCP$ and $\angle CFP = \angle CDP$, and since $PC = PD$ it follows that $\angle DFP = \angle CFP$. Therefore EF bisects $\angle CFD$.

(Proposed at IMO 1994)

8.7 The key to this problem is noticing that you can treat triangles ABD and ACE as completely separate, and ignore $\triangle ABC$. The only things these two triangles have in common is the angle at A and the height from A. Let these quantities be θ and h respectively. If we can express $\frac{1}{MB} + \frac{1}{MD}$ in terms of θ and h then we are done.

Let us rename D to C so that we are working with $\triangle ABC$ and can use the usual notation.

$$\begin{aligned}\frac{1}{MB} + \frac{1}{MC} &= \frac{1}{y} + \frac{1}{z} \\ &= \frac{y+z}{yz} \\ &= \frac{a}{yz} \\ &= \frac{ahrsx}{hrsxyz} \\ &= \frac{ahrsx}{hrK^2} \quad \text{(Heron's Formula)} \\ &= \frac{2K^2x}{hrK^2} \\ &= \frac{x}{r} \cdot \frac{2}{h} \\ &= \frac{2}{h} \cot \frac{\theta}{2}.\end{aligned}$$

(Proposed at IMO 1993)

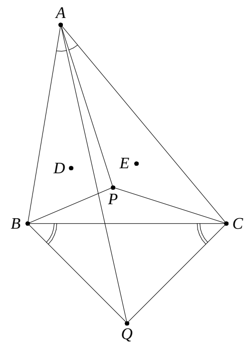
8.8 Construct Q so that $\angle BAQ = \angle PAC$ and $\angle ABQ = \angle APC$. Then by construction, $\triangle ABQ \,|||\, \triangle APC$. Now in $\triangle APB$ and $\triangle ACQ$:

- $\angle BAP = \angle BAC - \angle PAC = \angle QAC$

- $\frac{AC}{AQ} = \frac{AC}{AC \cdot AB/AP} = \frac{AP}{AB}$.

Hence $\triangle APB \,|||\, \triangle ACQ$. Now $\angle CBQ = \angle APC - \angle ABC = \angle APB - \angle ACB = \angle BCQ$, so $\triangle BCQ$ is isosceles. It follows that

$$\frac{AC}{PC} = \frac{AQ}{BQ} = \frac{AQ}{CQ} = \frac{AB}{BP}.$$



Now from the angle bisector theorem, BD will cut AP in the ratio $AB : BP$, and CE will cut AP in the ratio $AC : CP$. Since these ratios are the same, the three lines will be concurrent.

(IMO 1996 Question 2)

10 Recommended further reading

Geometric inequalities often require techniques from the world of standard inequalities. *Inequalities for the Olympiad Enthusiast*, by Graeme West (part of the same series as this booklet) provides some good material in this field.

This booklet is well under 100 pages, and as such cannot do proper justice to the rich field of classical geometry. A highly regarded and very readable reference is *Geometry Revisited*, by Coxeter and Greitzer.

A good source of problems are the yearbooks of the South African training program for the IMO (*South Africa and the* n*th IMO*, for $n \geq 35$). These contain problems and solutions for all the problems used in the training problem, including many good geometry problems.

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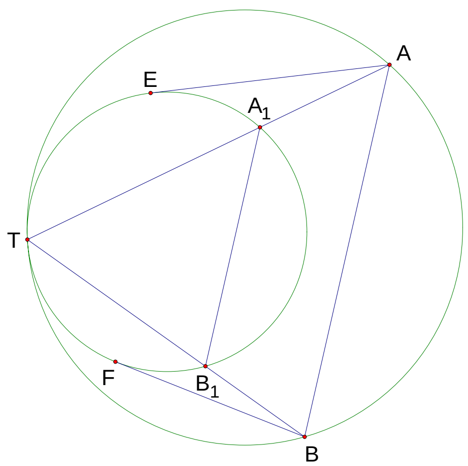
terug naar echt bestand

6.5 meetkundelemma's

A Metric Relation and its Applications

Son Hong Ta

Lemma. Let γ be a circle and let A and B be two arbitrary points on it. A circle ρ touches γ internally at T. Denote by AE and BF the tangent lines to ρ at E and F, respectively. Then $\frac{TA}{TB} = \frac{AE}{BF}$.



Proof. Denote by A_1 and B_1 the second intersections of TA and TB with ρ, respectively. We know that A_1B_1 is parallel to AB. Therefore,

$$\left(\frac{AE}{TA_1}\right)^2 = \frac{AA_1 \cdot AT}{A_1T \cdot A_1T} = \frac{BB_1}{B_1T} \cdot \frac{BT}{B_1T} = \left(\frac{BF}{TB_1}\right)^2.$$

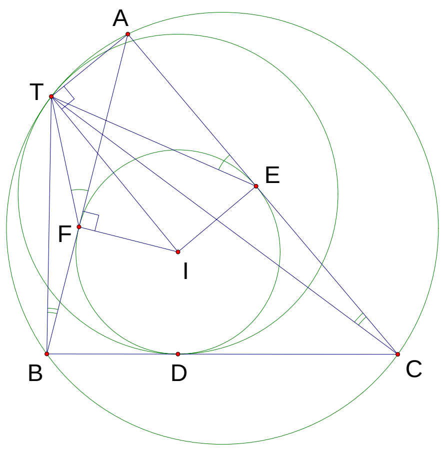
Hence,

$$\frac{AE}{TA_1} = \frac{BF}{TB_1} \implies \frac{AE}{BF} = \frac{TA_1}{TB_1} = \frac{TA}{TB},$$

which completes the proof. □

To illustrate how this lemma works, let us consider some examples. The following problem was proposed by Nguyen Minh Ha, in the Vietnamese Mathematics Magazine, in 2007.

Problem 1. Let Ω be the circumcircle of the triangle ABC and let D be the tangency point of its incircle $\rho(I)$ with the side BC. Let ω be the circle internally tangent to Ω at T, and to BC at D. Prove that $\angle ATI = 90^\circ$.



Solution. Let E and F be the tangency points of $\rho(I)$ with sides CA and AB, respectively. According to the lemma,

$$\frac{TB}{TC} = \frac{BD}{CD} = \frac{BF}{CE}.$$

Therefore triangles TBF and TCE are similar. It follows that $\measuredangle TFA = \measuredangle TEA$, hence the points A, I, E, F, T lie on the same circle. It follows that $\measuredangle ATI = \measuredangle AFI = 90^\circ$ which completes our proof. □

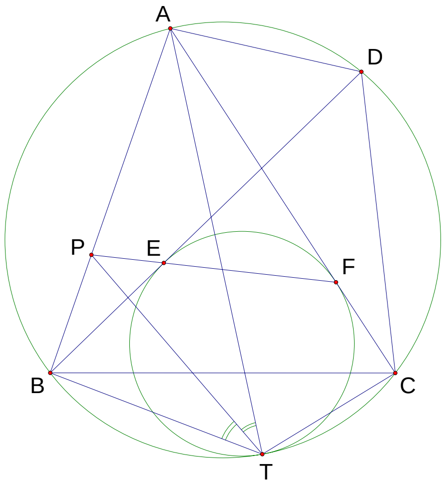
Problem 2. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let ω be a circle internally tangent to Ω at T, and to DB and AC at E and F, respectively. Let P be the intersection of EF and AB. Prove that TP is the internal angle bisector of the angle $\measuredangle ATB$.

Solution. From our lemma, applied to circles Ω, ω and points A, B, we conclude that $\frac{AT}{BT} = \frac{AF}{BE}$, thus it suffices to prove that

$$\frac{AF}{BE} = \frac{AP}{PB}.$$

Indeed, notice that $\measuredangle PEB = \measuredangle AFP$, and from the Law of Sines, applied to triangles APF, BPE, we have

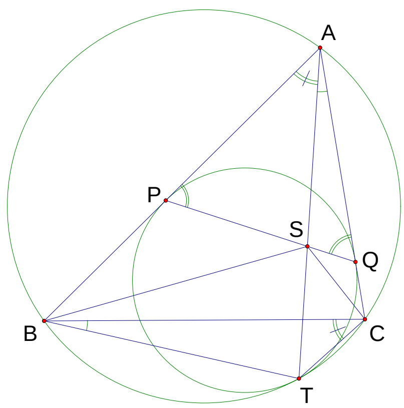
$$\frac{AP}{AF} = \frac{\sin \measuredangle AFP}{\sin \measuredangle APF} = \frac{\sin \measuredangle BEP}{\sin \measuredangle BPE} = \frac{BP}{BE}.$$



Therefore $\frac{AF}{BE} = \frac{AP}{PB}$, which completes our solution. $\square$

The third problem comes from the Moldovan Team Selection Test in 2007, which can be found in [2] and [3].

Problem 3. Let ABC be a triangle and let Ω be its circumcircle. Circles ω is internally tangent to Ω at T, and to sides AB and AC at P and Q, respectively. Let S be the intersection of AT and PQ. Prove that $\measuredangle SBA = \measuredangle SCA$.



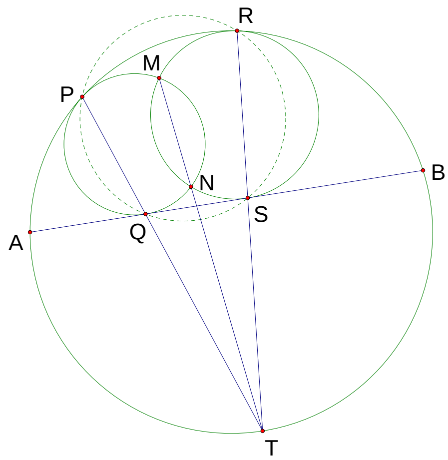
Solution. Using our lemma, we have

$$\frac{BP}{CQ} = \frac{BT}{CT} = \frac{\sin \measuredangle BCT}{\sin \measuredangle CBT} = \frac{\sin \measuredangle BAT}{\sin \measuredangle CAT} = \frac{PS}{QS}.$$

This fact implies that BPS and CQS are similar triangles which in turn implies that $\measuredangle SBA = \measuredangle SCA$. $\square$

Problem 4. Consider a circle (O) and a chord AB. Let circles (O_1), (O_2) be internally tangent to (O) and AB and let M and N their intersection. Prove that MN passes through the midpoint of the arc AB which does not contain M and N.

Solution. Denote by P and Q the tangency points of the circle (O_1) with (O) and AB, respectively. Let R and S be the tangency points of circle (O_2) with (O) and AB, respectively. Let T be the middle point of the arc AB which does not contain M and N.



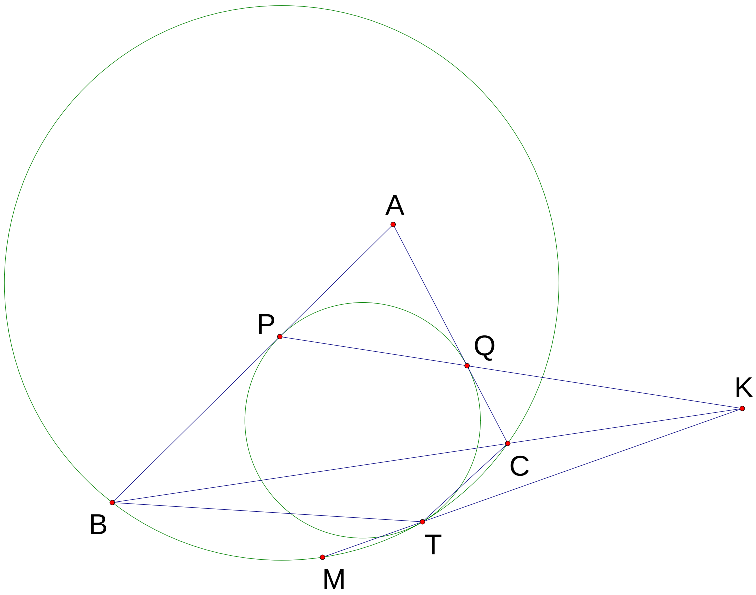
Applying the above lemma to circles (O), (O_1), and points A, B along with their tangent lines AQ, BQ to (O_1) we get $\frac{PA}{PB} = \frac{QA}{AB}$. This means that PQ passes through T. Similarly, RS passes through T. On the other hand, $\measuredangle PQA = \measuredangle QTA + \measuredangle QAT = \measuredangle PRA + \measuredangle ART = \measuredangle PRS$, therefore, points P, Q, R, S lie on a circle which we will denote by (O_3). We have that PQ is the radical axis of (O_1) and (O_3), RS is the radical axis of (O_2) and (O_3), and MN is the radical axis of (O_1) and (O_2). So, MN, PQ, and RS are concurrent at the radical center of the three circles. Hence, we deduce that MN passes through T, which is the midpoint of the arc AB that does not contain M and N. $\square$

We continue with a problem from the MOSP Tests 2007 [4].

Problem 5. Let ABC be a triangle. Circle ω passes through points B and C. Circle ω_1 is tangent internally to ω and also to the sides AB and AC at T, P, and Q, respectively. Let M be midpoint of arc BC (containing T) of ω. Prove that lines PQ, BC, and MT are concurrent.

Solution. Let $K = PQ \cap BC$ and let $K' = MT \cap BC$. Applying Menelaos' Theorem in triangle ABC we obtain

$$\frac{KB}{KB} \cdot \frac{QC}{QA} \cdot \frac{PA}{PB} = 1 \implies \frac{KB}{KC} = \frac{BP}{CQ}.$$



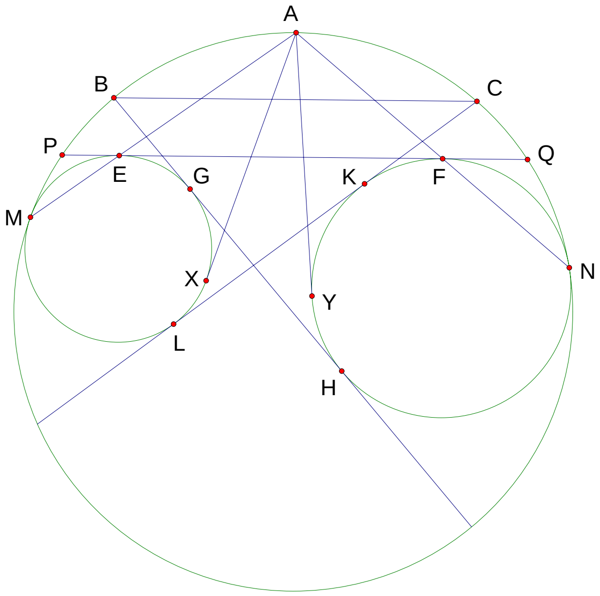
On the other hand, M is the midpoint of arc BC (containing T) of ω so MT is the external bisector of angle $\measuredangle BTC$, therefore $\frac{K'B}{K'C} = \frac{TB}{TC}$. Thus, we are left to prove that $\frac{BP}{CQ} = \frac{TB}{TC}$, which is true according to our lemma and we are done. □

The last problem was given in [5] and is also discussed and proved in [6]. Now, we will present another solution for this nice problem.

Problem 6. Circles (O_1) and (O_2) are internally tangent to a given circle (O) at M and N, respectively. Their internal common tangents intersect (O) at four points. Let B and C be two of them such that B and C lie on the same side with respect to O_1O_2. Prove that BC is parallel to an external common tangent of (O_1) and (O_2).

Solution. Draw the internal common tangents GH, KL of (O_1), (O_2) such that G and L lie on (O_1) and K and H lie on (O_2). Let EF be the external common tangent of (O_1), (O_2) such that E and B lie on the same side with respect to O_1O_2. Denote by P and Q the intersections of EF with (O). We will prove that BC is parallel to PQ. Denote by A be the midpoint of the arc PQ which does not contain M and N. Let AX and AY be the tangents at X and Y to the circles (O_1)

and (O_2). In the solution to Problem 4 we have proved that A, E, and M are collinear; A, F, and N are collinear, and the quadrilateral $MEFN$ is cyclic. Therefore, $AX^2 = AE \cdot AM = AF \cdot AN = AY^2$, i.e. $AX = AY$ (1).



Based on the lemma, $\frac{MA}{AX} = \frac{MB}{BG} = \frac{MC}{CL}$. On the other hand, by the Ptolemy's Theorem, $MA \cdot BC = MB \cdot AC = MC \cdot AB$, therefore

$$AX \cdot BC = BG \cdot AC = CL \cdot AB.$$

Similarly,

$$AY \cdot BC = BH \cdot AC + CK \cdot AB.$$

Thus $AC \cdot (BH - BG) = AB \cdot (CL - CK)$, i.e. $AC \cdot GH = AB \cdot KL$, which implies $AC = AB$. Hence, A is the midpoint of the arc BC of the circle (O). This means that BC is parallel to PQ and our solution is complete. □

References

[1] Mathlinks, *Nice geometry*, http://www.mathlinks.ro/viewtopic.php?t=170192

[2] Mathlinks, *A circle tangent to the circumcircle and two sides*, http://www.mathlinks.ro/viewtopic.php?t=140464

[3] Mathlinks, *Equal angle*, http://www.mathlinks.ro/Forum/viewtopic.php?t=98968

[4] 2007 Mathematical Olympiad Summer Program Tests, available at http://www.unl.edu/amc/a-activities/a6-mosp/archivemosp.shtml

[5] Shay Gueron, *Two Applications of the Generalized Ptolemy Theorem*, American Mathematical Monthly 2002.

[6] Mathlinks, *Parallel tangent*, http://www.mathlinks.ro/viewtopic.php?t=15945

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Lemmas in Euclidean Geometry[1]

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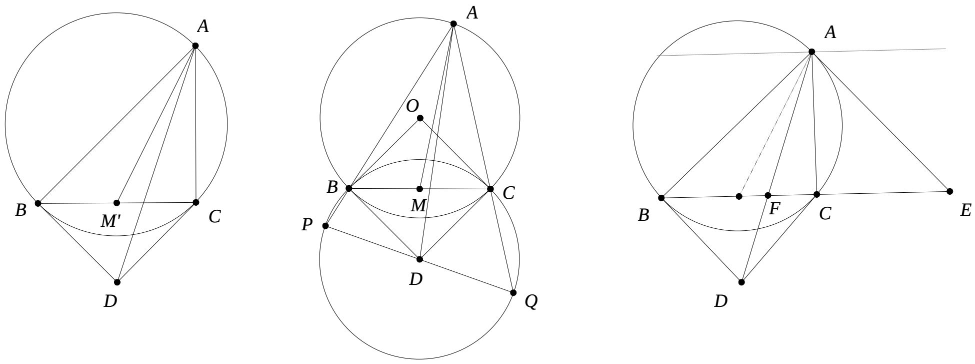
1. **Construction of the symmedian.**

Let ABC be a triangle and Γ its circumcircle. Let the tangent to Γ at B and C meet at D. Then AD coincides with a symmedian of $\triangle ABC$. (The *symmedian* is the reflection of the median across the angle bisector, all through the same vertex.)



We give three proofs. The first proof is a straightforward computation using Sine Law. The second proof uses similar triangles. The third proof uses projective geometry.

First proof. Let the reflection of AD across the angle bisector of $\angle BAC$ meet BC at M'. Then

$$\frac{BM'}{M'C} = \frac{AM'\frac{\sin\angle BAM'}{\sin\angle ABC}}{AM'\frac{\sin\angle CAM'}{\sin\angle ACB}} = \frac{\sin\angle BAM'}{\sin\angle ACD}\frac{\sin\angle ABD}{\sin\angle CAM'} = \frac{\sin\angle CAD}{\sin\angle ACD}\frac{\sin\angle ABD}{\sin\angle BAD} = \frac{CD}{AD}\frac{AD}{BD} = 1$$

Therefore, AM' is the median, and thus AD is the symmedian. □

Second proof. Let O be the circumcenter of ABC and let ω be the circle centered at D with radius DB. Let lines AB and AC meet ω at P and Q, respectively. Since $\angle PBQ = \angle DQC + \angle BAC = \frac{1}{2}(\angle BDC + \angle DOC) = 90^\circ$, we see that PQ is a diameter of ω and hence passes through D. Since $\angle ABC = \angle AQP$ and $\angle ACB = \angle APQ$, we see that triangles ABC and AQP are similar. If M is the midpoint of BC, noting that D is the midpoint of QP, the similarity implies that $\angle BAM = \angle QAD$, from which the result follows. □

Third proof. Let the tangent of Γ at A meet line BC at E. Then E is the pole of AD (since the polar of A is AE and the pole of D is BC). Let BC meet AD at F. Then point B, C, E, F are harmonic. This means that line AB, AC, AE, AF are harmonic. Consider the reflections of the four line across the angle bisector of $\angle BAC$. Their images must be harmonic too. It's easy to check that AE maps onto a line parallel to BC. Since BC must meet these four lines at harmonic points, it follows that the reflection of AF must pass through the midpoint of BC. Therefore, AF is a symmedian. □

[1]Updated July 26, 2008

Related problems:

(i) (Poland 2000) Let ABC be a triangle with $AC = BC$, and P a point inside the triangle such that $\angle PAB = \angle PBC$. If M is the midpoint of AB, then show that $\angle APM + \angle BPC = 180°$.

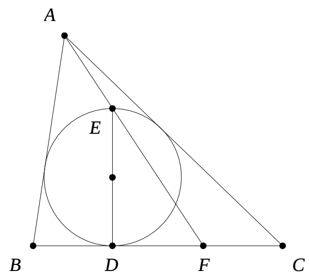
(ii) (IMO Shortlist 2003) Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose center does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

(iii) (Vietnam TST 2001) In the plane, two circles intersect at A and B, and a common tangent intersects the circles at P and Q. Let the tangents at P and Q to the circumcircle of triangle APQ intersect at S, and let H be the reflection of B across the line PQ. Prove that the points A, S, and H are collinear.

(iv) (USA TST 2007) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lies on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

(v) (USA 2008) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

2. **Diameter of the incircle.**



Let the incircle of triangle ABC touch side BC at D, and let DE be a diameter of the circle. If line AE meets BC at F, then $BD = CF$.

Proof. Consider the dilation with center A that carries the incircle to an excircle. The diameter DE of the incircle must be mapped to the diameter of the excircle that is perpendicular to BC. It follows that E must get mapped to the point of tangency between the excircle and BC. Since the image of E must lie on the line AE, it must be F. That is, the excircle is tangent to BC at F. Then, it follows easily that $BD = CF$. □

Related problems:

(i) (IMO Shortlist 2005) In a triangle ABC satisfying $AB+BC = 3AC$ the incircle has centre I and touches the sides AB and BC at D and E, respectively. Let K and L be the symmetric points of D and E with respect to I. Prove that the quadrilateral $ACKL$ is cyclic.

(ii) (IMO 1992) In the plane let $\mathcal{C}$ be a circle, ℓ a line tangent to the circle $\mathcal{C}$, and M a point on ℓ. Find the locus of all points P with the following property: there exists two points Q, R on ℓ such that M is the midpoint of QR and $\mathcal{C}$ is the inscribed circle of triangle PQR.

(iii) (USAMO 1999) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

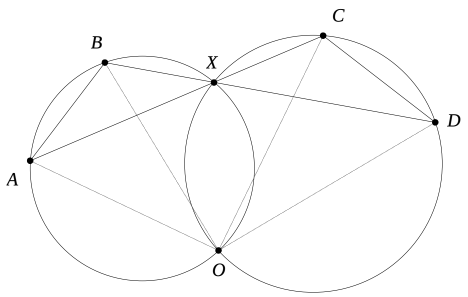
(iv) (USAMO 2001) Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

(v) (Tournament of Towns 2003 Fall) Triangle ABC has orthocenter H, incenter I and circumcenter O. Let K be the point where the incircle touches BC. If IO is parallel to BC, then prove that AO is parallel to HK.

(vi) (IMO 2008) Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

3. **Dude, where's my spiral center?**

Let AB and CD be two segments, and let lines AC and BD meet at X. Let the circumcircles of ABX and CDX meet again at O. Then O is the center of the spiral similarity that carries AB to CD.



Proof. Since $ABOX$ and $CDXO$ are cyclic, we have $\angle OBD = \angle OAC$ and $\angle OCA = \angle ODB$. It follows that triangles AOC and BOD are similar. The result is immediate. □

Remember that spiral similarities always come in pairs: if there is a spiral similarity that carries AB to CD, then there is one that carries AC to BD.

Related problems:

(i) (IMO Shortlist 2006) Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle CBA = \angle DCA = \angle EDA.$$

Diagonals BD and CE meet at P. Prove that line AP bisects side CD.

(ii) (China 1992) Convex quadrilateral $ABCD$ is inscribed in circle ω with center O. Diagonals AC and BD meet at P. The circumcircles of triangles ABP and CDP meet at P and Q. Assume that points O, P, and Q are distinct. Prove that $\angle OQP = 90°$.

(iii) Let $ABCD$ be a quadrilateral. Let diagonals AC and BD meet at P. Let O_1 and O_2 be the circumcenters of APD and BPC. Let M, N and O be the midpoints of AC, BD and O_1O_2. Show that O is the circumcenter of MPN.

(iv) (USAMO 2006) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

(v) (IMO 2005) Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

(vi) (IMO Shortlist 2002) Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

(vii) (USA TST 2006) In acute triangle ABC, segments AD, BE, and CF are its altitudes, and H is its orthocenter. Circle ω, centered at O, passes through A and H and intersects sides AB and AC again at Q and P (other than A), respectively. The circumcircle of triangle OPQ is tangent to segment BC at R. Prove that $CR/BR = ED/FD$.

(viii) (IMO Shortlist 2006) Points A_1, B_1 and C_1 are chosen on sides BC, CA, and AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, and CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, and C_2, respectively ($A_2 \neq A, B_2 \neq B$, and $C_2 \neq C$). Points A_3, B_3, and C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of sides BC, CA, and AB, respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

4. **Arc midpoints are equidistant to vertices and in/excenters**

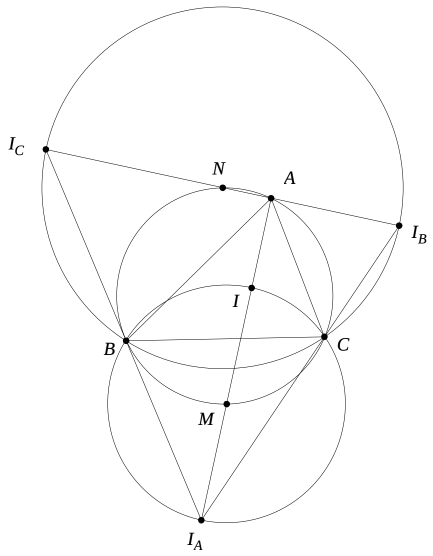
Let ABC be a triangle, I its incenter, and I_A, I_B, I_C its excenters. On the circumcircle of ABC, let M be the midpoint of the arc BC not containing A and let N be the midpoint of the arc BC containing A. Then $MB = MC = MI = MI_A$ and $NB = NC = NI_B = NI_C$.

Proof. Straightforward angle-chasing (do it yourself!). Another perspective is to consider the circumcircle of ABC as the nine-point-circle of $I_AI_BI_C$. □

Related problems:

(i) (APMO 2007) Let ABC be an acute angled triangle with $\angle BAC = 60°$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

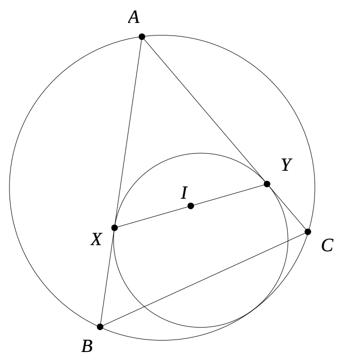
(ii) (IMO 2006) Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

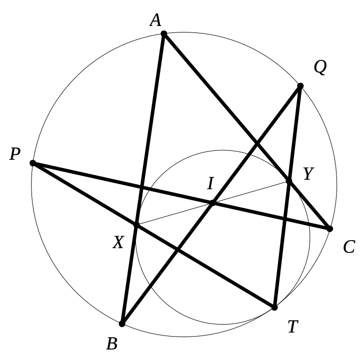


(iii) (Romanian TST 1996) Let $ABCD$ be a cyclic quadrilateral and let $\mathcal{M}$ be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (16 points in total). Prove that there are two sets $\mathcal{K}$ and $\mathcal{L}$ of four parallel lines each, such that every line in $\mathcal{K} \cup \mathcal{L}$ contains exactly four points of $\mathcal{M}$.

5. **I is the midpoint of the touch-chord of the mixtilinear incircles**

Let ABC be a triangle and I its incenter. Let Γ be the circle tangent to sides AB, AC, as well as the circumcircle of ABC. Let Γ touch AB and AC at X and Y, respectively. Then I is the midpoint of XY.





Proof. Let the point of tangency between the two circles be T. Extend TX and TY to meet the circumcircle of ABC again at P and Q respectively. Note that P and Q are the midpoint of the arcs AB and AC. Apply Pascal's theorem to $BACPTQ$ and we see that X, I, Y are collinear. Since I lies on the angle bisector of $\angle XAY$ and $AX = AY$, I must be the midpoint of XY. □

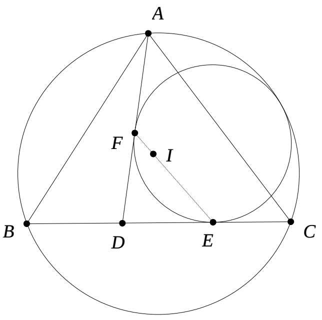
Related problems:

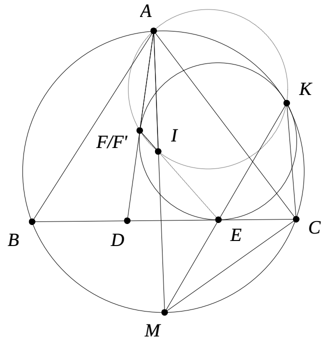
(i) (IMO 1978) In triangle ABC, $AB = AC$. A circle is tangent internally to the circumcircle of triangle ABC and also to sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC.

(ii) Let ABC be a triangle. Circle ω is tangent to AB and AC, and internally tangent to the circumcircle of triangle ABC. The circumcircle and ω are tangent at P. Let I be the incircle of triangle ABC. Line PI meets the circumcircle of ABC at P and Q. Prove that $BQ = CQ$.

6. **More curvilinear incircles.**

(A generalization of the previous lemma) Let ABC be a triangle, I its incenter and D a point on BC. Consider the circle that is tangent to the circumcircle of ABC but is also tangent to DC, DA at E, F respectively. Then E, F and I are collinear.





Proof. There is a "computational" proof using Casey's theorem[2] and transversal theorem[3]. You can try to work that out yourself. Here, we show a clever but difficult synthetic proof (communicated to me via Oleg Golberg).

Denote Ω the circumcircle of ABC and Γ the circle tangent tangent to the circumcircle of ABC and lines DC, DA. Let Ω and Γ touch at K. Let M be the midpoint of arc $\widehat{BC}$ on Ω not containing K. Then K, E, M are collinear (think: dilation with center K carrying Γ to Ω). Also, A, I, M are collinear, and $MI = MC$.

Let line EI meet Γ again at F'. It suffices to show that AF' is tangent to Γ.

Note that $\angle KF'E$ is subtended by $\widehat{KE}$ in Γ and $\angle KAM$ is subtended by $\widehat{KM}$ in Ω. Since $\widehat{KE}$ and $\widehat{KM}$ are homothetic with center K, we have $\angle KF'E = \angle KAM$, implying that A, K, I', F' are concyclic.

We have $\angle BCM = \angle CBM = \angle CKM$. So $\triangle MCE \sim \triangle MKC$. Hence $MC^2 = ME \cdot MK$. Since $MC = MI$, we have $MI^2 = ME \cdot MK$, implying that $\triangle MIE \sim \triangle MKI$. Therefore,

[2]**Casey's theorem**, also known as Generalized Ptolemy Theorem, states that if there are four circles $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ (could be degenerated into a point) all touching a circle Γ such that their tangency points follow that order around the circle, then

$$t_{12}t_{34} + t_{23}t_{14} = t_{13}t_{24},$$

where t_{12} is the length of the common tangent between Γ_i and Γ_j (if Γ_i and Γ_j on the same side of Γ, then take their common external tangent, else take their common internal tangent.) I think the converse is also true—if both equations hold, then there is some circle tangent to all four circles.

[3]The **transversal theorem** is a criterion for collinearity. It states that if A, B, C are three collinear points, and P is a point not on the line ABC, and A', B', C' are arbitrary points on lines PA, PB, PC respectively, then A', B', C' are collinear if and only if

$$BC \cdot \frac{AP}{A'P} + CA \cdot \frac{BP}{B'P} + AB \cdot \frac{CP}{C'P} = 0,$$

where the lengths are directed. In my opinion, it's much easier to remember the proof than to memorize this huge formula. The simplest derivation is based on relationships between the areas of $[PAB]$, $[PA'B']$, etc.

$\angle KEI = \angle AIK = \angle AF'K$ (since A, K, I, F' are concyclic). Therefore, AF' is tangent to Ω and the proof is complete. □

Related problems:

(i) (Bulgaria 2005) Consider two circles k_1, k_2 touching externally at point T. A line touches k_2 at point X and intersects k_1 at points A and B. Let S be the second intersection point of k_1 with the line XT . On the arc $\widehat{TS}$ not containing A and B is chosen a point C . Let CY be the tangent line to k_2 with $Y \in k_2$, such that the segment CY does not intersect the segment ST . If $I = XY \cap SC$. Prove that:

   (a) the points C, T, Y, I are concyclic.

   (b) I is the excenter of triangle ABC with respect to the side BC.

(ii) (Sawayama-Thébault[4]) Let ABC be a triangle with incenter I. Let D a point on side BC. Let P be the center of the circle that touches segments AD, DC, and the circumcircle of ABC, and let Q be the center of the circle that touches segments AD, BD, and the circumcircle of ABC. Show that P, Q, I are collinear.

(iii) Let P be a quadrilateral inscribed in a circle Ω, and let $\mathcal{Q}$ be the quadrilateral formed by the centers of the fourcircles internally touching O and each of the two diagonals of P. Show that the incenters of the four triangles having for sides the sides and diagonals of P form a rectangle R inscribed in Q.

(iv) (Romania 1997) Let ABC be a triangle with circumcircle Ω, and D a point on the side BC. Show that the circle tangent to Ω, AD and BD, and the circle tangent to Ω, AD and DC, are tangent to each other if and only if $\angle BAD = \angle CAD$.

(v) (Romania TST 2006) Let ABC be an acute triangle with $AB \neq AC$. Let D be the foot of the altitude from A and ω the circumcircle of the triangle. Let ω_1 be the circle tangent to AD, BD and ω. Let ω_2 be the circle tangent to AD, CD and ω. Let ℓ be the interior common tangent to both ω_1 and ω_2, different from CD. Prove that ℓ passes through the midpoint of BC if and only if $2BC = AB + AC$.

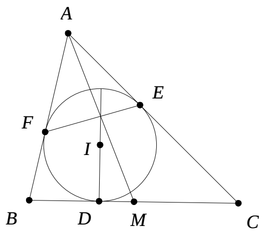
(vi) (AMM 10368) For each point O on diameter AB of a circle, perform the following construction. Let the perpendicular to AB at O meet the circle at point P. Inscribe circles in the figures bounded by the circle and the lines AB and OP. Let R and S be the points at which the two incircles to the curvilinear triangles AOP and BOP are tangent to the diameter AB. Show that $\angle RPS$ is independent of the position of O.

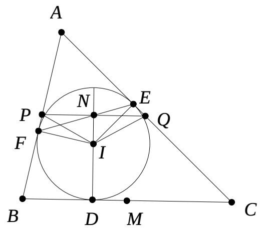
7. **Concurrent lines from the incircle.**

Let the incircle of ABC touch sides BC, CA, AB at D, E, F respectively. Let I be the incenter of ABC and M be the midpoint of BC. Then the lines EF, DI and AM are concurrent.

Proof. Let lines DI and EF meet at N. Construct a line through N parallel to BC, and let it meet sides AB and AC at P and Q, respectively. We need to show that A, N, M are collinear, so it suffices to show that N is the midpoint of PQ. We present two ways to finish this off, one using Simson's line, and the other using spiral similarities.

[4] A bit of history: this problem was posed by French geometer Victor Thébault (1882–1960) in the *American Mathematical Monthly* in 1938 (Problem 2887, 45 (1938) 482–483) and it remained unsolved until 1973. However, in 2003, Jean-Louis Ayme discovered that this problem was independently proposed and solved by instructor Y. Sawayama of the Central Military School of Tokyo in 1905! For more discussion, see Ayme's paper at `http://forumgeom.fau.edu/FG2003volume3/FG200325.pdf`





Simson line method: Consider the triangle APQ. The projections of the point I onto the three sides of APQ are D, N, F, which are collinear, I must lie on the circumcircle of APQ by Simson's theorem. But since AI is an angle bisector, $PI = QI$, thus $PN = QN$.

Spiral similarity method: Note that P, N, I, F are concyclic, so $\angle EFI = \angle QPI$. Similarly, $\angle PQI = \angle FEI$. So triangles PIQ and FIE are similar. Since $FI = EI$, we have $PI = QI$, and thus $PN = QN$. (c.f. Lemma 3) □

Related problems:

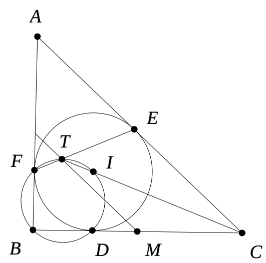
(i) (China 1999) In triangle ABC, $AB \neq AC$. Let D be the midpoint of side BC, and let E be a point on median AD. Let F be the foot of perpendicular from E to side BC, and let P be a point on segment EF. Let M and N be the feet of perpendiculars from P to sides AB and AC, respectively. Prove that M, E, and N are collinear if and only if $\angle BAP = \angle PAC$.

(ii) (IMO Shortlist 2005) The median AM of a triangle ABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y. The lines AX and AY intersect BC at P and Q. Prove that $BP = CQ$.

8. **More circles around the incircle.**

Let I be the incenter of triangle ABC, and let its incircle touch sides BC, AC, AB at D, E and F, respectively. Let line CI meet EF at T. Then T, I, D, B, F are concyclic. Consequent results include: $\angle BTC = 90°$, and T lies on the line connecting the midpoints of AB and BC.

An easier way to remember the third part of the lemma is: for a triangle ABC, draw a midline, an angle bisector, and a touch-chord, each generated from different vertex, then the three lines are concurrent.



Proof. Showing that I, T, E, B are concyclic is simply angle chasing (e.g. show that $\angle BIC = \angle BFE$). The second part follows from $\angle BTC = \angle BTI = \angle BFI = 90°$. For the third part, note that if M is the midpoint of BC, then M is the midpoint of an hypotenuse of the right triangle BTC. So $MT = MC$. Then $\angle MTC = \angle MCT = \angle ACT$, so MT is parallel to AC, and so MT is a midline of the triangle. □

Related problems:

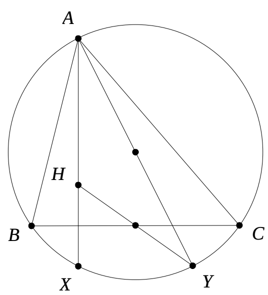
(i) Let ABC be an acute triangle whose incircle touches sides AC and AB at E and F, respectively. Let the angle bisectors of $\angle ABC$ and $\angle ACB$ meet EF at X and Y, respectively, and let the midpoint of BC be Z. Show that XYZ is equilateral if and only if $\angle A = 60°$.

(ii) (IMO Shortlist 2004) For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

(iii) Let points A and B lie on the circle Γ, and let C be a point inside the circle. Suppose that ω is a circle tangent to segments AC, BC and Γ. Let ω touch AC and Γ at P and Q. Show that the circumcircle of APQ passes through the incenter of ABC.

9. **Reflections of the orthocenter lie on the circumcircle.**

Let H be the orthocenter of triangle ABC. Let the reflection of H across the BC be X and the reflection of H across the midpoint of BC be Y. Then X and Y both lie on the circumcircle of ABC. Moreover, AY is a diameter of the circumcircle.



Proof. Trivial. Angle chasing. □

Related problems:

(i) Prove the existence of the nine-point circle. (Given a triangle, the nine-point circle is the circle that passes through the three midpoints of sides, the three feet of altitudes, and the three midpoints between the orthocenter and the vertices).

(ii) Let ABC be a triangle, and P a point on its circumcircle. Show that the reflections of P across the three sides of ABC lie on a lie that passes through the orthocenter of ABC.

(iii) (IMO Shortlist 2005) Let ABC be an acute-angled triangle with $AB \neq AC$, let H be its orthocentre and M the midpoint of BC. Points D on AB and E on AC are such that $AE = AD$ and D, H, E are collinear. Prove that HM is orthogonal to the common chord of the circumcircles of triangles ABC and ADE.

(iv) (USA TST 2005) Let $A_1A_2A_3$ be an acute triangle, and let O and H be its circumcenter and orthocenter, respectively. For $1 \le i \le 3$, points P_i and Q_i lie on lines OA_i and $A_{i+1}A_{i+2}$ (where $A_{i+3} = A_i$), respectively, such that OP_iHQ_i is a parallelogram. Prove that

$$\frac{OQ_1}{OP_1} + \frac{OQ_2}{OP_2} + \frac{OQ_3}{OP_3} \ge 3.$$

(v) (China TST quizzes 2006) Let ω be the circumcircle of triangle ABC, and let P be a point inside the triangle. Rays AP, BP, CP meet ω at A_1, B_1, C_1, respectively. Let A_2, B_2, C_2 be the images of A_1, B_1, C_1 under reflection about the midpoints of BC, CA, AB, respectively. Show that the orthocenter of ABC lies on the circumcircle of $A_2B_2C_2$.

10. **O and H are isogonal conjugates.**

Let ABC be a triangle, with circumcenter O, orthocenter H, and incenter I. Then AI is the angle bisector of $\angle HAO$.

Proof. Trivial. □

Related problems:

(i) (Crux) Points O and H are the circumcenter and orthocenter of acute triangle ABC, respectively. The perpendicular bisector of segment AH meets sides AB and AC at D and E, respectively. Prove that $\angle DOA = \angle EOA$.

(ii) Show that $IH = IO$ if and only if one of $\angle A$, $\angle B$, $\angle C$ is $60°$.

terug naar echt bestand

meetkundelemma's

6.6 projectieve meetkunde

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Projective Geometry

Milivoje Lukić

Contents

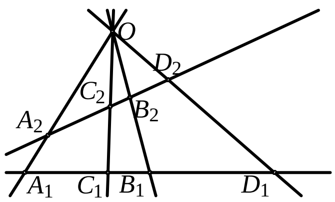
1 Cross Ratio. Harmonic Conjugates. Perspectivity. Projectivity

Definition 1. *Let A, B, C, and D be colinear points. The* cross ratio *of the pairs of points* (A,B) *and* (C,D) *is*

$$\mathscr{R}(A,B;C,D) = \frac{\overrightarrow{AC}}{\overrightarrow{CB}} : \frac{\overrightarrow{AD}}{\overrightarrow{DB}}. \quad (1)$$

Let a, b, c, d be four concurrent lines. For the given lines p_1 and p_2 let us denote $A_i = a \cap p_i$, $B_i = b \cap p_i$, $C_i = c \cap p_i$, $D_i = d \cap p_i$, for $i = 1, 2$. Then

$$\mathscr{R}(A_1,B_1;C_1,D_1) = \mathscr{R}(A_2,B_2;C_2,D_2). \quad (2)$$

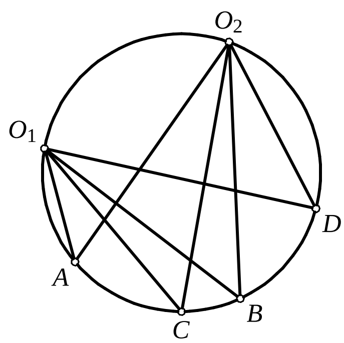


Thus it is meaningful to define the cross ratio of the pairs of concurrent points as

$$\mathscr{R}(a,b;c,d) = \mathscr{R}(A_1,B_1;C_1,D_1). \quad (3)$$

Assume that points O_1, O_2, A, B, C, D belong to a circle. Then

$$\begin{aligned} & \mathscr{R}(O_1A,O_1B;O_1C,O_1D) \\ = \; & \mathscr{R}(O_2A,O_2B;O_2C,O_2D). \end{aligned} \quad (4)$$

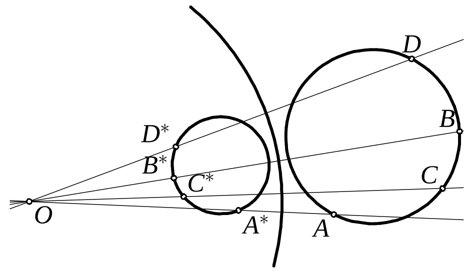


Hence it is meaningful to define the cross-ratio for cocyclic points as

$$\mathscr{R}(A,B;C,D) = \mathscr{R}(O_1A,O_1B;O_1C,O_1D). \quad (5)$$

Assume that the points A, B, C, D are colinear or cocyclic. Let an inversion with center O maps A, B, C, D into A^*, B^*, C^*, D^*. Then

$$\mathscr{R}(A,B;C,D) = \mathscr{R}(A^*,B^*;C^*,D^*). \qquad (6)$$



Definition 2. *Assume that A, B, C, and D are cocyclic or colinear points. Pairs of points* (A,B) *and* (C,D) *are* harmonic conjugates *if* $\mathscr{R}(A,B;C,D) = -1$. *We also write* $\mathscr{H}(A,B;C,D)$ *when we want to say that* (A,B) *and* (C,D) *are harmonic conjugates to each other.*

Definition 3. *Let each of* l_1 *and* l_2 *be either line or circle.* Perspectivity *with respect to the point S* $\overset{S}{\barwedge}$, *is the mapping of* $l_1 \to l_2$, *such that*

(i) If either l_1 *or* l_2 *is a circle than it contains S;*

(ii) every point $A_1 \in l_1$ *is mapped to the point* $A_2 = OA_1 \cap l_2$.

According to the previous statements perspectivity preserves the cross ratio and hence the harmonic conjugates.

Definition 4. *Let each of* l_1 *and* l_2 *be either line or circle.* Projectivity *is any mapping from* l_1 *to* l_2 *that can be represented as a finite composition of perspectivities.*

Theorem 1. *Assume that the points A, B, C,* D_1, *and* D_2 *are either colinear or cocyclic. If the equation* $\mathscr{R}(A,B;C,D_1) = \mathscr{R}(A,B;C,D_2)$ *is satisfied, then* $D_1 = D_2$. *In other words, a projectivity with three fixed points is the identity.*

Theorem 2. *If the points A, B, C, D are mutually discjoint and* $\mathscr{R}(A,B;C,D) = \mathscr{R}(B,A;C,D)$ *then* $\mathscr{H}(A,B;C,D)$.

2 Desargue's Theorem

The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are *perspective with respect to a center* if the lines A_1A_2, B_1B_2, and C_1C_2 are concurrent. They are *perspective with respect to an axis* if the points $K = B_1C_1 \cap B_2C_2$, $L = A_1C_1 \cap A_2C_2$, $M = A_1B_1 \cap A_2B_2$ are colinear.

Theorem 3 (Desargue). *Two triangles are perspective with respect to a center if and only if they are perspective with respect to a point.*

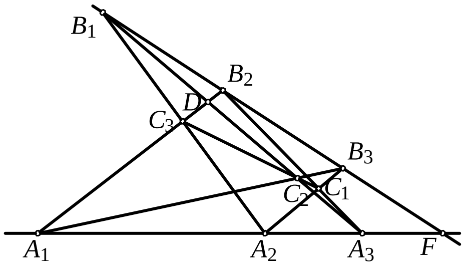
3 Theorems of Pappus and Pascal

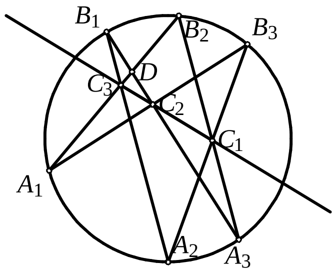
Theorem 4 (Pappus). *The points* A_1, A_2, A_3 *belong to the line a, and the points* B_1, B_2, B_3 *belong to the line b. Assume that* $A_1B_2 \cap A_2B_1 = C_3$, $A_1B_3 \cap A_3B_1 = C_2$, $A_2B_3 \cap A_3B_2 = C_1$. *Then* C_1, C_2, C_3 *are colinear.*

Proof. Denote $C_2' = C_1C_3 \cap A_3B_1$, $D = A_1B_2 \cap A_3B_1$, $E = A_2B_1 \cap A_3B_2$, $F = a \cap b$. Our goal is to prove that the points C_2 and C_2' are identical. Consider the sequence of projectivities:

$$A_3B_1DC_2 \overset{A_1}{\barwedge} FB_1B_2B_3 \overset{A_2}{\barwedge} A_3EB_2C_1 \overset{C_3}{\barwedge} A_3B_1DC_2'.$$

We have got the projective transformation of the line A_3B_1 that fixes the points A_3, B_1, D, and maps C_2 to C_2'. Since the projective mapping with three fixed points is the identity we have $C_2 = C_2'$. □





Theorem 5 (Pascal). *Assume that the points A_1, A_2, A_3, B_1, B_2, B_3 belong to a circle. The point in intersections of A_1B_2 with A_2B_1, A_1B_3 with A_3B_1, A_2B_3 with A_3B_2 lie on a line.*

Proof. The points C_2', D, and E as in the proof of the Pappus theorem. Consider the sequence of perspectivities

$$A_3B_1DC_2 \overset{A_1}{\barwedge} A_3B_1B_2B_3 \overset{A_2}{\barwedge} A_3EB_2C_1 \overset{C_3}{\barwedge} A_3B_1DC_2'.$$

In the same way as above we conclude that $C_2 = C_2'$. □

4 Pole. Polar. Theorems of Brianchon and Brokard

Definition 5. *Given a circle $k(O,r)$, let A^* be the image of the point $A \neq O$ under the inversion with respect to k. The line a passing through A^* and perpendicular to OA is called the* polar *of A with respect to k. Conversely A is called the* pole *of a with respect to k.*

Theorem 6. *Given a circle $k(O,r)$, let and a and b be the polars of A and B with respect to k. The $A \in b$ if and only if $B \in a$.*

Proof. $A \in b$ if and only if $\angle AB^*O = 90^\circ$. Analogously $B \in a$ if and only if $\angle BA^*O = 90^\circ$, and it reamins to notice that according to the basic properties of inversion we have $\angle AB^*O = \angle BA^*O$. □

Definition 6. *Points A and B are called* conjugated *with respect to the circle k if one of them lies on a polar of the other.*

Theorem 7. *If the line determined by two conjugated points A and B intersects $k(O,r)$ at C and D, then $\mathscr{H}(A,B;C,D)$. Conversely if $\mathscr{H}(A,B;C,D)$, where $C,D \in k$ then A and B are conjugated with respect to k.*

Proof. Let C_1 and D_1 be the intersection points of OA with k. Since the inversion preserves the cross-ratio and $\mathscr{R}(C_1,D_1;A,A^*) = \mathscr{R}(C_1,D_1;A^*,A)$ we have

$$\mathscr{H}(C_1,D_1;A,A^*). \tag{7}$$

Let p be the line that contains A and intersects k at C and D. Let $E = CC_1 \cap DD_1$, $F = CD_1 \cap DC_1$. Since C_1D_1 is the diameter of k we have $C_1F \perp D_1E$ and $D_1F \perp C_1E$, hence F is the orthocenter of the triangle C_1D_1E. Let $B = EF \cap CD$ and $\overline{A}^* = EF \cap C_1D_1$. Since

$$C_1D_1A\overline{A}^* \overset{E}{\barwedge} CDAB \overset{F}{\barwedge} D_1C_1A\overline{A}^*$$

have $\mathscr{H}(C_1,D_1;A,\overline{A}^*)$ and $\mathscr{H}(C,D;A,B)$. (7) now implies two facts:

1° From $\mathscr{H}(C_1,D_1;A,\overline{A}^*)$ and $\mathscr{H}(C_1,D_1;A,A^*)$ we get $A^* = \overline{A}^*$, hence $A^* \in EF$. However, since $EF \perp C_1D_1$, the line $EF = a$ is the polar of A.

2° For the point B which belongs to the polar of A we have $\mathscr{H}(C,D;A,B)$. This completes the proof. □

Theorem 8 (Brianchon's theorem). *Assume that the hexagon $A_1A_2A_3A_4A_5A_6$ is circumscribed about the circle k. The lines A_1A_4, A_2A_5, and A_3A_6 intersect at a point.*

Proof. We will use the convention in which the points will be denoted by capital latin letters, and their repsective polars with the corresponding lowercase letters.

Denote by M_i, $i=1,2,\dots,6$, the points of tangency of A_iA_{i+1} with k. Since $m_i = A_iA_{i+1}$, we have $M_i \in a_i$, $M_i \in a_{i+1}$, hence $a_i = M_{i-1}M_i$.

Let $b_j = A_jA_{j+3}$, $j=1,2,3$. Then $B_j = a_j \cap a_{j+3} = M_{j-1}M_j \cap M_{j+3}M_{j+4}$. We have to prove that there exists a point P such that $P \in b_1, b_2, b_3$, or analogously, that there is a line p such that $B_1, B_2, B_3 \in p$. In other words we have to prove that the points B_1, B_2, B_3 are colinear. However this immediately follows from the Pascal's theorem applied to $M_1M_3M_5M_4M_6M_2$. $\square$

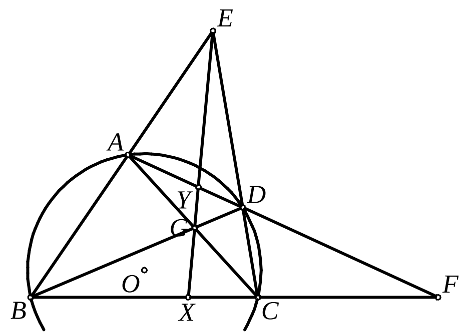
From the previous proof we see that the Brianchon's theorem is obtained from the Pascal's by replacing all the points with their polars and all lines by theirs poles.

Theorem 9 (Brokard). *The quadrilateral ABCD is inscribed in the circle k with center O. Let* $E = AB \cap CD$, $F = AD \cap BC$, $G = AC \cap BD$. *Then O is the orthocenter of the triangle EFG.*

Proof. We will prove that EG is a polar of F. Let $X = EG \cap BC$ and $Y = EG \cap AD$. Then we also have

$$ADYF \overset{E}{\barwedge} BCXF \overset{G}{\barwedge} DAYF,$$

which implies the relations $\mathscr{H}(A,D;Y,F)$ and $\mathscr{H}(B,C;X,F)$. According to the properties of polar we have that the points X and Y lie on a polar of the point F, hence EG is a polar of the point F.



Since EG is a polar of F, we have $EG \perp OF$. Analogously we have $FG \perp OE$, thus O is the orthocenter of $\triangle EFG$. $\square$

5 Problems

1. Given a quadrilateral $ABCD$, let $P = AB \cap CD$, $Q = AD \cap BC$, $R = AC \cap PQ$, $S = BD \cap PQ$. Prove that $\mathscr{H}(P,Q;R,S)$.

2. Given a triangle ABC and a point M on BC, let N be the point of the line BC such that $\angle MAN = 90^\circ$. Prove that $\mathscr{H}(B,C;M,N)$ if and only if AM is the bisector of the angle $\angle BAC$.

3. Let A and B be two points and let C be the point of the line AB. Using just a ruler find a point D on the line AB such that $\mathscr{H}(A,B;C,D)$.

4. Let A, B, C be the diagonal points of the quadrilateral $PQRS$, or equivalently $A = PQ \cap RS$, $B = QR \cap SP$, $C = PR \cap QS$. If only the points A, B, C, S, are given using just a ruler construct the points P, Q, R.

5. Assume that the incircle of $\triangle ABC$ touches the sides BC, AC, and AB at D, E, and F. Let M be the point such that the circle k_1 incscibed in $\triangle BCM$ touches BC at D, and the sides BM and CM at P and Q. Prove that the lines EF, PQ, BC are concurrent.

6. Given a triangle ABC, let D and E be the points on BC such that $BD = DE = EC$. The line p intersects AB, AD, AE, AC at K, L, M, N, respectively. Prove that $KN \geq 3LM$.

7. The point M_1 belongs to the side AB of the quadrilateral $ABCD$. Let M_2 be the projection of M_1 to the line BC from D, M_3 projection of M_2 to CD from A, M_4 projection of M_3 to DA from B, M_5 projection of M_4 to AB from C, etc. Prove that $M_{13} = M_1$.

8. (butterfly theorem) Points M and N belong to the circle k. Let P be the midpoint of the chord MN, and let AB and CD (A and C are on the same side of MN) be arbitrary chords of k passing through P. Prove that lines AD and BC intersect MN at points that are equidistant from P.

9. Given a triangle ABC, let D and E be the points of the sides AB and AC respectively such that $DE \| BC$. Let P be an interior point of the triangle ADE. Assume that the lines BP and CP intersect DE at F and G respectively. The circumcircles of $\triangle PDG$ and $\triangle PFE$ intersect at P and Q. Prove that the points A, P, and Q are colinear.

10. (IMO 1997 shortlist) Let $A_1A_2A_3$ be a non-isosceles triangle with the incenter I. Let C_i, $i = 1$, 2, 3, be the smaller circle through I tangent to both A_iA_{i+1} and A_iA_{i+2} (summation of indeces is done modulus 3). Let B_i, $i = 1$, 2, 3, be the other intersection point of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, A_3B_3I are colinear.

11. Given a triangle ABC and a point T, let P and Q be the feet of perpendiculars from T to the lines AB and AC, respectively. Let R and S be the feet of perpendiculars from A to TC and TB, respectively. Prove that the intersection of PR and QS belongs to BC.

12. Given a triangle ABC and a point M, a line passing through M intersects AB, BC, and CA at C_1, A_1, and B_1, respectively. The lines AM, BM, and CM intersect the circumcircle of $\triangle ABC$ repsectively at A_2, B_2, and C_2. Prove that the lines A_1A_2, B_1B_2, and C_1C_2 intersect in a point that belongs to the circumcircle of $\triangle ABC$.

13. Let P and Q isogonaly conjugated points and assume that $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$ are their pedal triangles, respectively. Let $X_1 = P_2Q_3 \cap P_3Q_2$, $X_2 = P_1Q_3 \cap P_3Q_1$, $X_3 = P_1Q_2 \cap P_2Q_1$. Prove that the points X_1, X_2, X_3 belong to the line PQ.

14. If the points A and M are conjugated with respect to k, then the circle with diameter AM is orthogonal to k.

15. From a point A in the exterior of a circle k two tangents AM and AN are drawn. Assume that K and L are two points of k such that A, K, L are colinear. Prove that MN bisects the segment PQ.

16. The point isogonaly conjugated to the centroid is called the *Lemuan* point. The lines connected the vertices with the Lemuan point are called *symmedians*. Assume that the tangents from B and C to the circumcircle Γ of $\triangle ABC$ intersect at the point P. Prove that AP is a symmedian of $\triangle ABC$.

17. Given a triangle ABC, assume that the incircle touches the sides BC, CA, AB at the points M, N, P, respectively. Prove that AM, BN, and CP intersect in a point.

18. Let $ABCD$ be a quadrilateral circumscribed about a circle. Let M, N, P, and Q be the points of tangency of the incircle with the sides AB, BC, CD, and DA respectively. Prove that the lines AC, BD, MP, and NQ intersect in a point.

19. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD intersect at O; extensions of the sides AB and CD at E; the tangents to the circumcircle from A and D at K; and the tangents to the circumcircle at B and C at L. Prove that the points E, K, O, and L lie on a line.

20. Let $ABCD$ be a cyclic quadrilateral. The lines AB and CD intersect at the point E, and the diagonals AC and BD at the point F. The circumcircle of the triangles $\triangle AFD$ and $\triangle BFC$ intersect again at H. Prove that $\angle EHF = 90^\circ$.

6 Solutions

1. Let $T = AC \cap BD$. Consider the sequence of the perspectivities
$$PQRS \overset{A}{\barwedge} BDTS \overset{C}{\barwedge} QPRS.$$
Since the perspectivity preserves the cross-ratio $\mathscr{R}(P,Q;R,S) = \mathscr{R}(Q,P;R,S)$ we obtain that $\mathscr{H}(P,Q;R,S)$.

2. Let $\alpha = \angle BAC$, $\beta = \angle CBA$, $\gamma = \angle ACB$ and $\varphi = \angle BAM$. Using the sine theorem on $\triangle ABM$ and $\triangle ACM$ we get
$$\frac{BM}{MC} = \frac{BM}{AM}\frac{AM}{CM} = \frac{\sin\varphi}{\sin\beta}\,\frac{\sin\gamma}{\sin(\alpha-\varphi)}.$$
Similarly using the sine theorem on $\triangle ABN$ and $\triangle ACN$ we get
$$\frac{BN}{NC} = \frac{BN}{AN}\frac{AN}{CN} = \frac{\sin(90^\circ-\varphi)}{\sin(180^\circ-\beta)}\,\frac{\sin\gamma}{\sin(90^\circ+\alpha-\varphi)}.$$
Combining the previous two equations we get
$$\frac{BM}{MC} : \frac{BN}{NC} = \frac{\tan\varphi}{\tan(\alpha-\varphi)}.$$
Hence, $|\mathscr{R}(B,C;M,N)| = 1$ is equivalent to $\tan\varphi = \tan(\alpha-\varphi)$, i.e. to $\varphi = \alpha/2$. Since $B \neq C$ and $M \neq N$, the relation $|\mathscr{R}(B,C;M,N)| = 1$ is equivalent to $\mathscr{R}(B,C;M,N) = -1$, and the statement is now shown.

3. The motivation is the problem 1. Choose a point K outside AB and point L on AK different from A and K. Let $M = BL \cap CK$ and $N = BK \cap AM$. Now let us construct a point D as $D = AB \cap LN$. From the problem 1 we indeed have $\mathscr{H}(A,B;C,D)$.

4. Let us denote $D = AS \cap BC$. According to the problem 1 we have $\mathscr{H}(R,S;A,D)$. Now we construct the point $D = AS \cap BC$. We have the points A, D, and S, hence according to the previous problem we can construct a point R such that $\mathscr{H}(A,D;S,R)$. Now we construct $P = BS \cap CR$ and $Q = CS \cap BR$, which solves the problem.

5. It is well known (and is easy to prove using Ceva's theorem) that the lines AD, BE, and CF intersect at a point G (called a Gergonne point of $\triangle ABC$) Let $X = BC \cap EF$. As in the problem 1 we have $\mathscr{H}(B,C;D,X)$. If we denote $X' = BC \cap PQ$ we analogously have $\mathscr{H}(B,C;D,X')$, hence $X = X'$.

6. Let us denote $x = KL$, $y = LM$, $z = MN$. We have to prove that $x+y+z \geq 3y$, or equivalently $x+z \geq 2y$. Since $\mathscr{R}(K,N;L,M) = \mathscr{R}(B,C;D,E)$, we have
$$\frac{x}{y+z} : \frac{x+y}{z} = \frac{\overrightarrow{KL}}{\overrightarrow{LN}} : \frac{\overrightarrow{KM}}{\overrightarrow{MN}} = \frac{\overrightarrow{BD}}{\overrightarrow{DC}} : \frac{\overrightarrow{BE}}{\overrightarrow{EC}} = \frac{1}{2} : \frac{1}{2},$$
implying $4xz = (x+y)(y+z)$.

   If it were $y > (x+z)/2$ we would have
$$x+y > \frac{3}{2}x + \frac{1}{2}z = 2\frac{1}{4}(x+x+x+z) \geq 2\sqrt[4]{xxxz},$$
and analogously $y+z > 2\sqrt[4]{xzzz}$ as well as $(x+y)(y+z) > 4xz$ which is a contradiction. Hence the assumption $y > (x+z)/2$ was false so we have $y \leq (x+z)/2$.

   Let us analyze the case of equality. If $y = (x+z)/2$, then $4xz = (x+y)(x+z) = (3x+z)(x+3z)/4$, which is equivalent to $(x-z)^2 = 0$. Hence the equality holds if $x = y = z$. We leave to the reader to prove that $x = y = z$ is satisfied if and only if $p \parallel BC$.

7. Let $E = AB \cap CD$, $F = AD \cap BC$. Consider the sequence of perspectivities

$$ABEM_1 \overset{D}{\barwedge} FBCM_2 \overset{A}{\barwedge} DECM_3 \overset{B}{\barwedge} DAFM_4 \overset{C}{\barwedge} EABM_5. \tag{8}$$

According to the conditions given in the problem this sequence of perspectivites has two be applied three more times to arrive to the point M_{13}. Notice that the given sequence of perspectivities maps A to E, E to B, and B to A. Clearly if we apply (8) three times the points A, B, and E will be fixed while M_1 will be mapped to M_{13}. Thus $M_1 = M_{13}$.

8. Let X' be the point symmetric to Y with respect to P. Notice that

$$\begin{aligned}
\mathscr{R}(M,N;X,P) &= \mathscr{R}(M,N;P,Y) \quad (\text{from } MNXP \overset{D}{\barwedge} MNAC \overset{B}{\barwedge} MNPY)\\
&= \mathscr{R}(N,M;P,X') \quad (\text{ the reflection with the center } P \text{ preserves}\\
&\qquad\qquad\qquad\qquad\quad \text{the ratio, hence it preserves the cross-ratio})\\
&= \frac{1}{\mathscr{R}(N,M;X',P)} = \mathscr{R}(M,N;X',P),
\end{aligned}$$

where the last equality follows from the basic properties of the cross ratio. It follows that $X = X'$.

9. Let $J = DQ \cap BP$, $K = EQ \cap CP$. If we prove that $JK \parallel DE$ this would imply that the triangles BDJ and CEK are perspective with the respect to a center, hence with repsect to an axis as well (according to Desargue's theorem) which immediately implies that A, P, Q are colinear (we encourage the reader to verify this fact).

   Now we will prove that $JK \| DE$. Let us denote $T = DE \cap PQ$. Applying the Menelaus theorem on the triangle DTQ and the line PF we get

$$\frac{\overrightarrow{DJ}}{\overrightarrow{JQ}} \frac{\overrightarrow{QP}}{\overrightarrow{PT}} \frac{\overrightarrow{TF}}{\overrightarrow{FD}} = -1.$$

   Similarly from the triangle ETQ and the line PG:

$$\frac{\overrightarrow{EK}}{\overrightarrow{KQ}} \frac{\overrightarrow{QP}}{\overrightarrow{PT}} \frac{\overrightarrow{TG}}{\overrightarrow{GE}} = -1.$$

   Dividing the last two equalities and using $DT \cdot TG = FT \cdot TE$ (T is on the radical axis of the circumcircles of $\triangle DPG$ and $\triangle FPE$), we get

$$\frac{\overrightarrow{DJ}}{\overrightarrow{JQ}} = \frac{\overrightarrow{EK}}{\overrightarrow{KQ}}.$$

   Thus $JK \parallel DE$, q.e.d.

10. Apply the inversion with the respect to I. We leave to the reader to draw the inverse picture. Notice that the condition that I is the incentar now reads that the circumcircles $A_i^* A_{i+1}^* I$ are of the same radii. Indeed if R is the radius of the circle of inversion and r the distance between I and XY then the radius of the circumcircle of $\triangle IX^*Y^*$ is equal to R^2/r. Now we use the following statement that is very easy to prove: "Let k_1, k_2, k_3 be three circles such that all pass through the same point I, but no two of them are mutually tangent. Then the centers of these circles are colinear if and only if there exists another common point $J \neq I$ of these three circles."

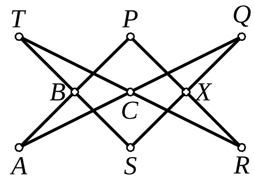
    In the inverse picture this transforms into proving that the lines $A_1^*B_1^*$, $A_2^*B_2^*$, and $A_3^*B_3^*$ intersect at a point.

In order to prove this it is enough to show that the corresponding sides of the triangles $A_1^*A_2^*A_3^*$ and $B_1^*B_2^*B_3^*$ are parallel (then these triangles would be perspective with respect to the infinitely far line). Afterwards the Desargue's theorem would imply that the triangles are perspective with respect to a center. Let P_i^* be the incenter of $A_{i+1}^*A_{i+2}^*I$, and let Q_i^* be the foot of the perpendicular from I to $P_{i+1}^*P_{i+2}^*$. It is easy to prove that

$$\overrightarrow{A_1^*A_2^*} = 2\overrightarrow{Q_1^*Q_2^*} = -\overrightarrow{P_1^*P_2^*}.$$

Also since the circles $A_i^*A_{i+1}^*I$ are of the same radii, we have $P_1^*P_2^* \parallel B_1^*B_2^*$, hence $A_1^*A_2^* \parallel B_1^*B_2^*$.

11. We will prove that the intersection X of PR and QS lies on the line BC. Notice that the points P, Q, R, S belong to the circle with center AT. Consider the six points A, S, R, T, P, Q that lie on a circle. Using Pascal's theorem with respect to the diagram



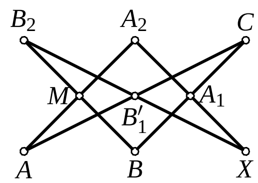
we get that the points B, C, and $X = PR \cap QS$ are colinear.

12. *First solution, using projective mappings.* Let $A_3 = AM \cap BC$ and $B_3 = BM \cap AC$. Let X be the other intersection point of the line A_1A_2 with the circumcircle k of $\triangle ABC$. Let X' be the other intersection point of the line B_1B_2 with k. Consider the sequence of perspectivities

$$ABCX \overset{A_2}{\overline{\wedge}} A_3BCA_1 \overset{M}{\overline{\wedge}} AB_3CB_1 \overset{B_2}{\overline{\wedge}} ABCX'$$

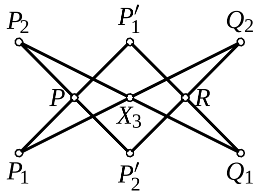
which has three fixed points A, B, C, hence $X = X'$. Analogously the line C_1C_2 contains X and the problem is completely solved.

Second solution, using Pascal's theorem. Assume that the line A_1A_2 intersect the circumcircle of the trianlge ABC at A_2 and X. Let $XB_2 \cap AC = B_1'$. Let us apply the Pascal's theorem on the points A, B, C, A_2, B_2, X according the diagram:



It follows that the points A_1, B_1', and M are colinear. Hence $B_1' \in A_1M$. According to the definition of the point B_1' we have $B_1' \in AC$ hence $B_1' = A_1M \cap AC = B_1$. The conclusion is that the points X, B_1, B_2 are colinear. Analogously we prove that the points X, C_1, C_2 are colinear, hence the lines A_1A_2, B_1B_2, C_1C_2 intersect at X that belongs to the circumcircle of the triangle ABC.

13. It is well known (from the theory of pedal triangles) that pedal triangles corresponding to the isogonally conjugated points have the common circumcircle, so called *pedal circle* of the points P and Q. The center of that circle which is at the same time the midpoint of PQ will be denoted by R. Let $P_1' = PP_1 \cap Q_1R$ and $P_2' = PP_2 \cap Q_2R$ (the points P_1' and P_2' belong to the pedal circle of the point P, as point on the same diameters as Q_1 and Q_2 respectively). Using the Pascal's theorem on the points Q_1, P_2, P_2', Q_2, P_1, P_1' in the order shown by the diagram



we get that the points P, R, X_1 are colinear or $X_1 \in PQ$. Analogously the points X_2, X_3 belong to the line PQ.

14. Let us recall the statement according to which the circle l is invariant under the inversion with respect to the circle k if and only if $l = k$ or $l \perp k$.

    Since the point M belongs to the polar of the point A with respect to k we have $\angle MA^*A = 90^\circ$ where $A^* = \psi_l(A)$. Therefore $A^* \in l$ where l is the circle with the radius AM. Analogously $M^* \in l$. However from $A \in l$ we get $A^* \in l^*$; $A^* \in l$ yields $A \in l^*$ (the inversion is inverse to itself) hence $\psi_l(A^*) = A$). Similarly we get $M \in l^*$ and $M^* \in l^*$. Notice that the circles l and l^* have the four common points A, A^*, M, M^*, which is exactly two too much. Hence $l = l^*$ and according to the statement mentioned at the beginning we conclude $l = k$ or $l \perp k$. The case $l = k$ can be easily eliminated, because the circle l has the diameter AM, and AM can't be the diameter of k because A and M are conjugated to each other.

    Thus $l \perp k$, q.e.d.

15. Let $J = KL \cap MN$, $R = l \cap MN$, $X_\infty = l \cap AM$. Since MN is the polar of A from $J \in MN$ we get $\mathscr{H}(K,L;J,A)$. From $KLJA \overset{M}{\overline{\wedge}} PQRX_\infty$ we also have $\mathscr{H}(P,Q;R,X_\infty)$. This implies that R is the midpoint of PQ.

16. Let Q be the intersection point of the lines AP and BC. Let Q' be the point of BC such that the ray AQ' is isogonal to the ray AQ in the triangle ABC. This exactly means that $\angle Q'AC = \angle BAQ$ i $\angle BAQ' = \angle QAC$.

    For an arbitrary point X of the segment BC, the sine theorem applied to triangles BAX and XAC yields

$$\frac{BX}{XC} = \frac{BX}{AX}\frac{AX}{XC} = \frac{\sin\angle BAX}{\sin\angle ABX}\frac{\sin\angle ACX}{\sin\angle XAC} = \frac{\sin\angle ACX}{\sin\angle ABX}\frac{\sin\angle BAX}{\sin\angle XAC} = \frac{AB}{AC}\frac{\sin\angle BAX}{\sin\angle XAC}.$$

    Applying this to $X = Q$ and $X = Q'$ and multiplying together afterwards we get

$$\frac{BQ}{QC}\frac{BQ'}{Q'C} = \frac{AB}{AC}\frac{\sin\angle BAQ}{\sin\angle QAC}\frac{AB}{AC}\frac{\sin\angle BAQ'}{\sin\angle Q'AC} = \frac{AB^2}{AC^2}. \tag{9}$$

    Hence if we prove $BQ/QC = AB^2/AC^2$ we would immediately have $BQ'/Q'C = 1$, making Q' the midpoint of BC. Then the line AQ is isogonaly conjugated to the median, implying the required statement.

    Since P belongs to the polars of B and C, then the points B and C belong to the polar of the point P, and we conclude that the polar of P is precisely BC. Consider the intersection D of the line BC with the tangent to the circumcircle at A. Since the point D belongs to the polars of A and P, AP has to be the polar of D. Hence $\mathscr{H}(B,C;D,Q)$. Let us now calculate the ratio BD/DC. Since the triangles ABD and CAD are similar we have $BD/AD = AD/CD = AB/AC$. This implies $BD/CD = (BD/AD)(AD/CD) = AB^2/AC^2$. The relation $\mathscr{H}(B,C;D,Q)$ implies $BQ/QC = BD/DC = AB^2/AC^2$, which proves the statement.

17. The statement follows from the Brianchon's theorem applied to $APBMCN$.

18. Applying the Brianchon's theorem to the hexagon $AMBCPD$ we get that the line MP contains the intersection of AB and CD. Analogously, applying the Brianchon's theorem to $ABNCDQ$ we get that NQ contains the same point.

19. The Brokard's theorem claims that the polar of $F = AD \cap BC$ is the line $f = EO$. Since the polar of the point on the circle is equal to the tangent at that point we know that $K = a \cap d$, where a and d are polars of the points A and D. Thus $k = AD$. Since $F \in AD = k$, we have $K \in f$ as well. Analogously we can prove that $L \in f$, hence the points E, O, K, L all belong to f.

20. Let $G = AD \cap BC$. Let k be the circumcircle of $ABCD$. Denote by k_1 and k_2 respectively the circumcircles of $\triangle ADF$ and $\triangle BCF$. Notice that AD is the radical axis of the circles k and k_1; BC the radical axis of k and k_2; and FH the radical axis of k_1 and k_2. According to the famous theorem these three radical axes intersect at one point G. In other words we have shown that the points F, G, H are colinear.

    Without loss of generality assume that F is between G and H (alternatively, we could use the oriented angles). Using the inscribed quadrilaterals $ADFH$ and $BCFH$, we get $\angle DHF = \angle DAF = \angle DAC$ and $\angle FHC = \angle FBC = \angle DBC$, hence $\angle DHC = \angle DHF + \angle FHC = \angle DAC + \angle DBC = 2\angle DAC = \angle DOC$. Thus the points D, C, H, and O lie on a circle. Similarly we prove that the points A, B, H, O lie on a circle.

    Denote by k_3 and k_4 respectively the circles circumscribed about the quadrilaterals $ABHO$ and $DCHO$. Notice that the line AB is the radical axis of the circles k and k_3. Simlarly CD and OH, respectively, are those of the pairs of circles (k, k_2) and (k_3, k_4). Thus these lines have to intersect at one point, and that has to be E. This proves that the points O, H, and E are colinear.

    According to the Brocard's theorem we have $FH \perp OE$, which according to $FH = GH$ and $OE = HE$ in turn implies that $GH \perp HE$, q.e.d.

terug naar echt bestand

6.7 projectieve meetkunde

Projective Geometry

Milivoje Lukić

Contents

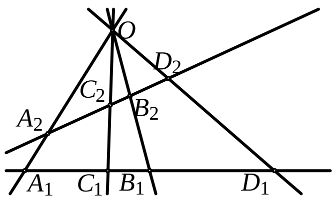
1 Cross Ratio. Harmonic Conjugates. Perspectivity. Projectivity

Definition 1. *Let A, B, C, and D be colinear points. The* cross ratio *of the pairs of points (A,B) and (C,D) is*

$$\mathscr{R}(A,B;C,D) = \frac{\overrightarrow{AC}}{\overrightarrow{CB}} : \frac{\overrightarrow{AD}}{\overrightarrow{DB}}. \qquad (1)$$

Let a, b, c, d be four concurrent lines. For the given lines p_1 and p_2 let us denote $A_i = a \cap p_i$, $B_i = b \cap p_i$, $C_i = c \cap p_i$, $D_i = d \cap p_i$, for $i = 1$, 2. Then

$$\mathscr{R}(A_1,B_1;C_1,D_1) = \mathscr{R}(A_2,B_2;C_2,D_2). \qquad (2)$$

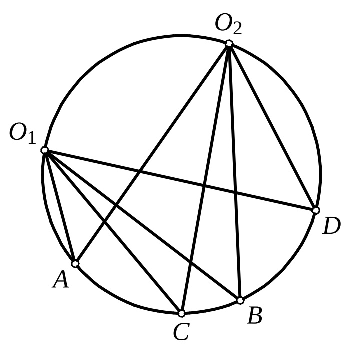


Thus it is meaningful to define the cross ratio of the pairs of concurrent points as

$$\mathscr{R}(a,b;c,d) = \mathscr{R}(A_1,B_1;C_1,D_1). \qquad (3)$$

Assume that points O_1, O_2, A, B, C, D belong to a circle. Then

$$\begin{aligned} & \mathscr{R}(O_1A,O_1B;O_1C,O_1D) \\ = \; & \mathscr{R}(O_2A,O_2B;O_2C,O_2D). \end{aligned} \qquad (4)$$

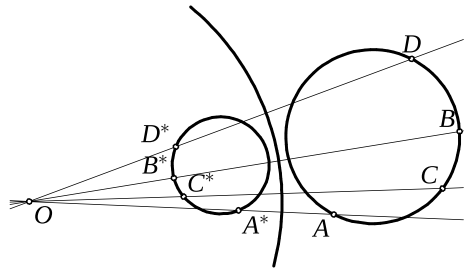


Hence it is meaningful to define the cross-ratio for cocyclic points as

$$\mathscr{R}(A,B;C,D) = \mathscr{R}(O_1A,O_1B;O_1C,O_1D). \qquad (5)$$

Assume that the points A, B, C, D are colinear or cocyclic. Let an inversion with center O maps A, B, C, D into A^*, B^*, C^*, D^*. Then

$$\mathscr{R}(A,B;C,D) = \mathscr{R}(A^*,B^*;C^*,D^*). \qquad (6)$$



Definition 2. *Assume that A, B, C, and D are cocyclic or colinear points. Pairs of points (A,B) and (C,D) are* harmonic conjugates *if $\mathscr{R}(A,B;C,D) = -1$. We also write $\mathscr{H}(A,B;C,D)$ when we want to say that (A,B) and (C,D) are harmonic conjugates to each other.*

Definition 3. *Let each of l_1 and l_2 be either line or circle.* Perspectivity *with respect to the point S $\overset{S}{\barwedge}$, is the mapping of $l_1 \to l_2$, such that*

(i) If either l_1 or l_2 is a circle than it contains S;

(ii) every point $A_1 \in l_1$ is mapped to the point $A_2 = OA_1 \cap l_2$.

According to the previous statements perspectivity preserves the cross ratio and hence the harmonic conjugates.

Definition 4. *Let each of l_1 and l_2 be either line or circle.* Projectivity *is any mapping from l_1 to l_2 that can be represented as a finite composition of perspectivities.*

Theorem 1. *Assume that the points A, B, C, D_1, and D_2 are either colinear or cocyclic. If the equation $\mathscr{R}(A,B;C,D_1) = \mathscr{R}(A,B;C,D_2)$ is satisfied, then $D_1 = D_2$. In other words, a projectivity with three fixed points is the identity.*

Theorem 2. *If the points A, B, C, D are mutually discjoint and $\mathscr{R}(A,B;C,D) = \mathscr{R}(B,A;C,D)$ then $\mathscr{H}(A,B;C,D)$.*

2 Desargue's Theorem

The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are *perspective with respect to a center* if the lines A_1A_2, B_1B_2, and C_1C_2 are concurrent. They are *perspective with respect to an axis* if the points $K = B_1C_1 \cap B_2C_2$, $L = A_1C_1 \cap A_2C_2$, $M = A_1B_1 \cap A_2B_2$ are colinear.

Theorem 3 (Desargue). *Two triangles are perspective with respect to a center if and only if they are perspective with respect to a point.*

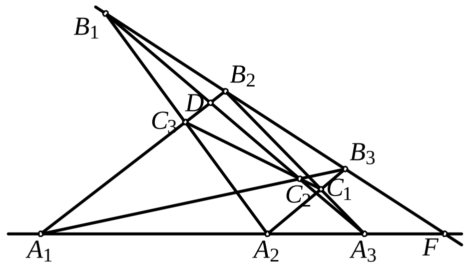
3 Theorems of Pappus and Pascal

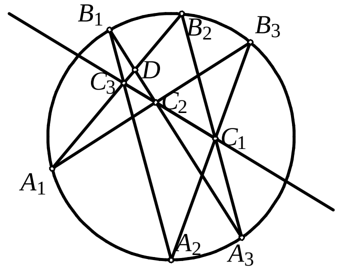
Theorem 4 (Pappus). *The points A_1, A_2, A_3 belong to the line a, and the points B_1, B_2, B_3 belong to the line b. Assume that $A_1B_2 \cap A_2B_1 = C_3$, $A_1B_3 \cap A_3B_1 = C_2$, $A_2B_3 \cap A_3B_2 = C_1$. Then C_1, C_2, C_3 are colinear.*

Proof. Denote $C_2' = C_1C_3 \cap A_3B_1$, $D = A_1B_2 \cap A_3B_1$, $E = A_2B_1 \cap A_3B_2$, $F = a \cap b$. Our goal is to prove that the points C_2 and C_2' are identical. Consider the sequence of projectivities:

$$A_3B_1DC_2 \overset{A_1}{\barwedge} FB_1B_2B_3 \overset{A_2}{\barwedge} A_3EB_2C_1 \overset{C_3}{\barwedge} A_3B_1DC_2'.$$

We have got the projective transformation of the line A_3B_1 that fixes the points A_3, B_1, D, and maps C_2 to C_2'. Since the projective mapping with three fixed points is the identity we have $C_2 = C_2'$. $\square$





Theorem 5 (Pascal). *Assume that the points A_1, A_2, A_3, B_1, B_2, B_3 belong to a circle. The point in intersections of A_1B_2 with A_2B_1, A_1B_3 with A_3B_1, A_2B_3 with A_3B_2 lie on a line.*

Proof. The points C_2', D, and E as in the proof of the Pappus theorem. Consider the sequence of perspectivities

$$A_3B_1DC_2 \overset{A_1}{\barwedge} A_3B_1B_2B_3 \overset{A_2}{\barwedge} A_3EB_2C_1 \overset{C_3}{\barwedge} A_3B_1DC_2'.$$

In the same way as above we conclude that $C_2 = C_2'$. □

4 Pole. Polar. Theorems of Brianchon and Brokard

Definition 5. *Given a circle $k(O,r)$, let A^* be the image of the point $A \neq O$ under the inversion with respect to k. The line a passing through A^* and perpendicular to OA is called the* polar *of A with respect to k. Conversely A is called the* pole *of a with respect to k.*

Theorem 6. *Given a circle $k(O,r)$, let and a and b be the polars of A and B with respect to k. The $A \in b$ if and only if $B \in a$.*

Proof. $A \in b$ if and only if $\angle AB^*O = 90^\circ$. Analogously $B \in a$ if and only if $\angle BA^*O = 90^\circ$, and it reamins to notice that according to the basic properties of inversion we have $\angle AB^*O = \angle BA^*O$. □

Definition 6. *Points A and B are called* conjugated *with respect to the circle k if one of them lies on a polar of the other.*

Theorem 7. *If the line determined by two conjugated points A and B intersects $k(O,r)$ at C and D, then $\mathscr{H}(A,B;C,D)$. Conversely if $\mathscr{H}(A,B;C,D)$, where $C,D \in k$ then A and B are conjugated with respect to k.*

Proof. Let C_1 and D_1 be the intersection points of OA with k. Since the inversion preserves the cross-ratio and $\mathscr{R}(C_1,D_1;A,A^*) = \mathscr{R}(C_1,D_1;A^*,A)$ we have

$$\mathscr{H}(C_1,D_1;A,A^*). \tag{7}$$

Let p be the line that contains A and intersects k at C and D. Let $E = CC_1 \cap DD_1$, $F = CD_1 \cap DC_1$. Since C_1D_1 is the diameter of k we have $C_1F \perp D_1E$ and $D_1F \perp C_1E$, hence F is the orthocenter of the triangle C_1D_1E. Let $B = EF \cap CD$ and $\overline{A}^* = EF \cap C_1D_1$. Since

$$C_1D_1A\overline{A}^* \overset{E}{\barwedge} CDAB \overset{F}{\barwedge} D_1C_1A\overline{A}^*$$

have $\mathscr{H}(C_1,D_1;A,\overline{A}^*)$ and $\mathscr{H}(C,D;A,B)$. (7) now implies two facts:

1° From $\mathscr{H}(C_1,D_1;A,\overline{A}^*)$ and $\mathscr{H}(C_1,D_1;A,A^*)$ we get $A^* = \overline{A}^*$, hence $A^* \in EF$. However, since $EF \perp C_1D_1$, the line $EF = a$ is the polar of A.

2° For the point B which belongs to the polar of A we have $\mathscr{H}(C,D;A,B)$. This completes the proof. □

Theorem 8 (Brianchon's theorem). *Assume that the hexagon $A_1A_2A_3A_4A_5A_6$ is circumscribed about the circle k. The lines A_1A_4, A_2A_5, and A_3A_6 intersect at a point.*

Proof. We will use the convention in which the points will be denoted by capital latin letters, and their repsective polars with the corresponding lowercase letters.

Denote by M_i, $i = 1, 2, \dots, 6$, the points of tangency of A_iA_{i+1} with k. Since $m_i = A_iA_{i+1}$, we have $M_i \in a_i$, $M_i \in a_{i+1}$, hence $a_i = M_{i-1}M_i$.

Let $b_j = A_jA_{j+3}$, $j = 1, 2, 3$. Then $B_j = a_j \cap a_{j+3} = M_{j-1}M_j \cap M_{j+3}M_{j+4}$. We have to prove that there exists a point P such that $P \in b_1, b_2, b_3$, or analogously, that there is a line p such that $B_1, B_2, B_3 \in p$. In other words we have to prove that the points B_1, B_2, B_3 are colinear. However this immediately follows from the Pascal's theorem applied to $M_1M_3M_5M_4M_6M_2$. □

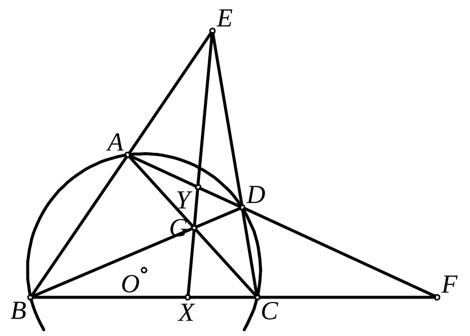
From the previous proof we see that the Brianchon's theorem is obtained from the Pascal's by replacing all the points with their polars and all lines by theirs poles.

Theorem 9 (Brokard). *The quadrilateral $ABCD$ is inscribed in the circle k with center O. Let $E = AB \cap CD$, $F = AD \cap BC$, $G = AC \cap BD$. Then O is the orthocenter of the triangle EFG.*

Proof. We will prove that EG is a polar of F. Let $X = EG \cap BC$ and $Y = EG \cap AD$. Then we also have

$$ADYF \overset{E}{\barwedge} BCXF \overset{G}{\barwedge} DAYF,$$

which implies the relations $\mathscr{H}(A, D; Y, F)$ and $\mathscr{H}(B, C; X, F)$. According to the properties of polar we have that the points X and Y lie on a polar of the point F, hence EG is a polar of the point F.



Since EG is a polar of F, we have $EG \perp OF$. Analogously we have $FG \perp OE$, thus O is the orthocenter of $\triangle EFG$. □

5 Problems

1. Given a quadrilateral $ABCD$, let $P = AB \cap CD$, $Q = AD \cap BC$, $R = AC \cap PQ$, $S = BD \cap PQ$. Prove that $\mathscr{H}(P, Q; R, S)$.

2. Given a triangle ABC and a point M on BC, let N be the point of the line BC such that $\angle MAN = 90^\circ$. Prove that $\mathscr{H}(B, C; M, N)$ if and only if AM is the bisector of the angle $\angle BAC$.

3. Let A and B be two points and let C be the point of the line AB. Using just a ruler find a point D on the line AB such that $\mathscr{H}(A, B; C, D)$.

4. Let A, B, C be the diagonal points of the quadrilateral $PQRS$, or equivalently $A = PQ \cap RS$, $B = QR \cap SP$, $C = PR \cap QS$. If only the points A, B, C, S, are given using just a ruler construct the points P, Q, R.

5. Assume that the incircle of $\triangle ABC$ touches the sides BC, AC, and AB at D, E, and F. Let M be the point such that the circle k_1 incscibed in $\triangle BCM$ touches BC at D, and the sides BM and CM at P and Q. Prove that the lines EF, PQ, BC are concurrent.

6. Given a triangle ABC, let D and E be the points on BC such that $BD = DE = EC$. The line p intersects AB, AD, AE, AC at K, L, M, N, respectively. Prove that $KN \geq 3LM$.

7. The point M_1 belongs to the side AB of the quadrilateral $ABCD$. Let M_2 be the projection of M_1 to the line BC from D, M_3 projection of M_2 to CD from A, M_4 projection of M_3 to DA from B, M_5 projection of M_4 to AB from C, etc. Prove that $M_{13} = M_1$.

8. (butterfly theorem) Points M and N belong to the circle k. Let P be the midpoint of the chord MN, and let AB and CD (A and C are on the same side of MN) be arbitrary chords of k passing through P. Prove that lines AD and BC intersect MN at points that are equidistant from P.

9. Given a triangle ABC, let D and E be the points of the sides AB and AC respectively such that $DE \| BC$. Let P be an interior point of the triangle ADE. Assume that the lines BP and CP intersect DE at F and G respectively. The circumcircles of $\triangle PDG$ and $\triangle PFE$ intersect at P and Q. Prove that the points A, P, and Q are colinear.

10. (IMO 1997 shortlist) Let $A_1A_2A_3$ be a non-isosceles triangle with the incenter I. Let C_i, $i = 1$, 2, 3, be the smaller circle through I tangent to both A_iA_{i+1} and A_iA_{i+2} (summation of indeces is done modulus 3). Let B_i, $i = 1$, 2, 3, be the other intersection point of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, A_3B_3I are colinear.

11. Given a triangle ABC and a point T, let P and Q be the feet of perpendiculars from T to the lines AB and AC, respectively. Let R and S be the feet of perpendiculars from A to TC and TB, respectively. Prove that the intersection of PR and QS belongs to BC.

12. Given a triangle ABC and a point M, a line passing through M intersects AB, BC, and CA at C_1, A_1, and B_1, respectively. The lines AM, BM, and CM intersect the circumcircle of $\triangle ABC$ repsectively at A_2, B_2, and C_2. Prove that the lines A_1A_2, B_1B_2, and C_1C_2 intersect in a point that belongs to the circumcircle of $\triangle ABC$.

13. Let P and Q isogonaly conjugated points and assume that $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$ are their pedal triangles, respectively. Let $X_1 = P_2Q_3 \cap P_3Q_2$, $X_2 = P_1Q_3 \cap P_3Q_1$, $X_3 = P_1Q_2 \cap P_2Q_1$. Prove that the points X_1, X_2, X_3 belong to the line PQ.

14. If the points A and M are conjugated with respect to k, then the circle with diameter AM is orthogonal to k.

15. From a point A in the exterior of a circle k two tangents AM and AN are drawn. Assume that K and L are two points of k such that A, K, L are colinear. Prove that MN bisects the segment PQ.

16. The point isogonaly conjugated to the centroid is called the *Lemuan* point. The lines connected the vertices with the Lemuan point are called *symmedians*. Assume that the tangents from B and C to the circumcircle Γ of $\triangle ABC$ intersect at the point P. Prove that AP is a symmedian of $\triangle ABC$.

17. Given a triangle ABC, assume that the incircle touches the sides BC, CA, AB at the points M, N, P, respectively. Prove that AM, BN, and CP intersect in a point.

18. Let $ABCD$ be a quadrilateral circumscribed about a circle. Let M, N, P, and Q be the points of tangency of the incircle with the sides AB, BC, CD, and DA respectively. Prove that the lines AC, BD, MP, and NQ intersect in a point.

19. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD intersect at O; extensions of the sides AB and CD at E; the tangents to the circumcircle from A and D at K; and the tangents to the circumcircle at B and C at L. Prove that the points E, K, O, and L lie on a line.

20. Let $ABCD$ be a cyclic quadrilateral. The lines AB and CD intersect at the point E, and the diagonals AC and BD at the point F. The circumcircle of the triangles $\triangle AFD$ and $\triangle BFC$ intersect again at H. Prove that $\angle EHF = 90°$.

6 Solutions

1. Let $T = AC \cap BD$. Consider the sequence of the perspectivities
$$PQRS \overset{A}{\barwedge} BDTS \overset{C}{\barwedge} QPRS.$$
Since the perspectivity preserves the cross-ratio $\mathscr{R}(P,Q;R,S) = \mathscr{R}(Q,P;R,S)$ we obtain that $\mathscr{H}(P,Q;R,S)$.

2. Let $\alpha = \angle BAC$, $\beta = \angle CBA$, $\gamma = \angle ACB$ and $\varphi = \angle BAM$. Using the sine theorem on $\triangle ABM$ and $\triangle ACM$ we get
$$\frac{BM}{MC} = \frac{BM}{AM}\frac{AM}{CM} = \frac{\sin\varphi}{\sin\beta}\,\frac{\sin\gamma}{\sin(\alpha-\varphi)}.$$
Similarly using the sine theorem on $\triangle ABN$ and $\triangle ACN$ we get
$$\frac{BN}{NC} = \frac{BN}{AN}\frac{AN}{CN} = \frac{\sin(90^\circ-\varphi)}{\sin(180^\circ-\beta)}\,\frac{\sin\gamma}{\sin(90^\circ+\alpha-\varphi)}.$$
Combining the previous two equations we get
$$\frac{BM}{MC} : \frac{BN}{NC} = \frac{\tan\varphi}{\tan(\alpha-\varphi)}.$$
Hence, $|\mathscr{R}(B,C;M,N)| = 1$ is equivalent to $\tan\varphi = \tan(\alpha-\varphi)$, i.e. to $\varphi = \alpha/2$. Since $B \neq C$ and $M \neq N$, the relation $|\mathscr{R}(B,C;M,N)| = 1$ is equivalent to $\mathscr{R}(B,C;M,N) = -1$, and the statement is now shown.

3. The motivation is the problem 1. Choose a point K outside AB and point L on AK different from A and K. Let $M = BL \cap CK$ and $N = BK \cap AM$. Now let us construct a point D as $D = AB \cap LN$. From the problem 1 we indeed have $\mathscr{H}(A,B;C,D)$.

4. Let us denote $D = AS \cap BC$. According to the problem 1 we have $\mathscr{H}(R,S;A,D)$. Now we construct the point $D = AS \cap BC$. We have the points A, D, and S, hence according to the previous problem we can construct a point R such that $\mathscr{H}(A,D;S,R)$. Now we construct $P = BS \cap CR$ and $Q = CS \cap BR$, which solves the problem.

5. It is well known (and is easy to prove using Ceva's theorem) that the lines AD, BE, and CF intersect at a point G (called a Gergonne point of $\triangle ABC$) Let $X = BC \cap EF$. As in the problem 1 we have $\mathscr{H}(B,C;D,X)$. If we denote $X' = BC \cap PQ$ we analogously have $\mathscr{H}(B,C;D,X')$, hence $X = X'$.

6. Let us denote $x = KL$, $y = LM$, $z = MN$. We have to prove that $x+y+z \geq 3y$, or equivalently $x+z \geq 2y$. Since $\mathscr{R}(K,N;L,M) = \mathscr{R}(B,C;D,E)$, we have
$$\frac{x}{y+z} : \frac{x+y}{z} = \frac{\overrightarrow{KL}}{\overrightarrow{LN}} : \frac{\overrightarrow{KM}}{\overrightarrow{MN}} = \frac{\overrightarrow{BD}}{\overrightarrow{DC}} : \frac{\overrightarrow{BE}}{\overrightarrow{EC}} = \frac{1}{2} : \frac{1}{2},$$
implying $4xz = (x+y)(y+z)$.

If it were $y > (x+z)/2$ we would have
$$x+y > \frac{3}{2}x + \frac{1}{2}z = 2\frac{1}{4}(x+x+x+z) \geq 2\sqrt[4]{xxxz},$$
and analogously $y+z > 2\sqrt[4]{xzzz}$ as well as $(x+y)(y+z) > 4xz$ which is a contradiction. Hence the assumption $y > (x+z)/2$ was false so we have $y \leq (x+z)/2$.

Let us analyze the case of equality. If $y = (x+z)/2$, then $4xz = (x+y)(x+z) = (3x+z)(x+3z)/4$, which is equivalent to $(x-z)^2 = 0$. Hence the equality holds if $x = y = z$. We leave to the reader to prove that $x = y = z$ is satisfied if and only if $p \parallel BC$.

7. Let $E = AB \cap CD$, $F = AD \cap BC$. Consider the sequence of perspectivities

$$ABEM_1 \overset{D}{\barwedge} FBCM_2 \overset{A}{\barwedge} DECM_3 \overset{B}{\barwedge} DAFM_4 \overset{C}{\barwedge} EABM_5. \qquad (8)$$

According to the conditions given in the problem this sequence of perspectivites has two be applied three more times to arrive to the point M_{13}. Notice that the given sequence of perspectivities maps A to E, E to B, and B to A. Clearly if we apply (8) three times the points A, B, and E will be fixed while M_1 will be mapped to M_{13}. Thus $M_1 = M_{13}$.

8. Let X' be the point symmetric to Y with respect to P. Notice that

$$\begin{aligned}
\mathscr{R}(M,N;X,P) &= \mathscr{R}(M,N;P,Y) \quad (\text{from } MNXP \overset{D}{\barwedge} MNAC \overset{B}{\barwedge} MNPY) \\
&= \mathscr{R}(N,M;P,X') \quad (\text{ the reflection with the center } P \text{ preserves} \\
&\qquad\qquad\qquad\qquad\quad \text{the ratio, hence it preserves the cross-ratio}) \\
&= \frac{1}{\mathscr{R}(N,M;X',P)} = \mathscr{R}(M,N;X',P),
\end{aligned}$$

where the last equality follows from the basic properties of the cross ratio. It follows that $X = X'$.

9. Let $J = DQ \cap BP$, $K = EQ \cap CP$. If we prove that $JK \parallel DE$ this would imply that the triangles BDJ and CEK are perspective with the respect to a center, hence with repsect to an axis as well (according to Desargue's theorem) which immediately implies that A, P, Q are colinear (we encourage the reader to verify this fact).

   Now we will prove that $JK \| DE$. Let us denote $T = DE \cap PQ$. Applying the Menelaus theorem on the triangle DTQ and the line PF we get

$$\frac{\overrightarrow{DJ}}{\overrightarrow{JQ}} \frac{\overrightarrow{QP}}{\overrightarrow{PT}} \frac{\overrightarrow{TF}}{\overrightarrow{FD}} = -1.$$

   Similarly from the triangle ETQ and the line PG:

$$\frac{\overrightarrow{EK}}{\overrightarrow{KQ}} \frac{\overrightarrow{QP}}{\overrightarrow{PT}} \frac{\overrightarrow{TG}}{\overrightarrow{GE}} = -1.$$

   Dividing the last two equalities and using $DT \cdot TG = FT \cdot TE$ (T is on the radical axis of the circumcircles of $\triangle DPG$ and $\triangle FPE$), we get

$$\frac{\overrightarrow{DJ}}{\overrightarrow{JQ}} = \frac{\overrightarrow{EK}}{\overrightarrow{KQ}}.$$

   Thus $JK \parallel DE$, q.e.d.

10. Apply the inversion with the respect to I. We leave to the reader to draw the inverse picture. Notice that the condition that I is the incentar now reads that the circumcircles $A_i^* A_{i+1}^* I$ are of the same radii. Indeed if R is the radius of the circle of inversion and r the distance between I and XY then the radius of the circumcircle of $\triangle IX^*Y^*$ is equal to R^2/r. Now we use the following statement that is very easy to prove: "Let k_1, k_2, k_3 be three circles such that all pass through the same point I, but no two of them are mutually tangent. Then the centers of these circles are colinear if and only if there exists another common point $J \neq I$ of these three circles."

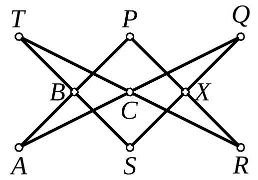
    In the inverse picture this transforms into proving that the lines $A_1^*B_1^*$, $A_2^*B_2^*$, and $A_3^*B_3^*$ intersect at a point.

In order to prove this it is enough to show that the corresponding sides of the triangles $A_1^*A_2^*A_3^*$ and $B_1^*B_2^*B_3^*$ are parallel (then these triangles would be perspective with respect to the infinitely far line). Afterwards the Desargue's theorem would imply that the triangles are perspective with respect to a center. Let P_i^* be the incenter of $A_{i+1}^*A_{i+2}^*I$, and let Q_i^* be the foot of the perpendicular from I to $P_{i+1}^*P_{i+2}^*$. It is easy to prove that

$$\overrightarrow{A_1^*A_2^*} = 2\overrightarrow{Q_1^*Q_2^*} = -\overrightarrow{P_1^*P_2^*}.$$

Also since the circles $A_i^*A_{i+1}^*I$ are of the same radii, we have $P_1^*P_2^* \parallel B_1^*B_2^*$, hence $A_1^*A_2^* \parallel B_1^*B_2^*$.

11. We will prove that the intersection X of PR and QS lies on the line BC. Notice that the points P, Q, R, S belong to the circle with center AT. Consider the six points A, S, R, T, P, Q that lie on a circle. Using Pascal's theorem with respect to the diagram



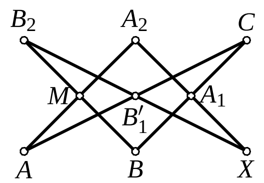
we get that the points B, C, and $X = PR \cap QS$ are colinear.

12. *First solution, using projective mappings.* Let $A_3 = AM \cap BC$ and $B_3 = BM \cap AC$. Let X be the other intersection point of the line A_1A_2 with the circumcircle k of $\triangle ABC$. Let X' be the other intersection point of the line B_1B_2 with k. Consider the sequence of perspectivities

$$ABCX \overset{A_2}{\overline{\wedge}} A_3BCA_1 \overset{M}{\overline{\wedge}} AB_3CB_1 \overset{B_2}{\overline{\wedge}} ABCX'$$

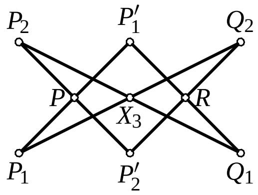
which has three fixed points A, B, C, hence $X = X'$. Analogously the line C_1C_2 contains X and the problem is completely solved.

Second solution, using Pascal's theorem. Assume that the line A_1A_2 intersect the circumcircle of the trianlge ABC at A_2 and X. Let $XB_2 \cap AC = B_1'$. Let us apply the Pascal's theorem on the points A, B, C, A_2, B_2, X according the diagram:



It follows that the points A_1, B_1', and M are colinear. Hence $B_1' \in A_1M$. According to the definition of the point B_1' we have $B_1' \in AC$ hence $B_1' = A_1M \cap AC = B_1$. The conclusion is that the points X, B_1, B_2 are colinear. Analogously we prove that the points X, C_1, C_2 are colinear, hence the lines A_1A_2, B_1B_2, C_1C_2 intersect at X that belongs to the circumcircle of the triangle ABC.

13. It is well known (from the theory of pedal triangles) that pedal triangles corresponding to the isogonally conjugated points have the common circumcircle, so called *pedal circle* of the points P and Q. The center of that circle which is at the same time the midpoint of PQ will be denoted by R. Let $P_1' = PP_1 \cap Q_1R$ and $P_2' = PP_2 \cap Q_2R$ (the points P_1' and P_2' belong to the pedal circle of the point P, as point on the same diameters as Q_1 and Q_2 respectively). Using the Pascal's theorem on the points Q_1, P_2, P_2', Q_2, P_1, P_1' in the order shown by the diagram



we get that the points P, R, X_1 are colinear or $X_1 \in PQ$. Analogously the points X_2, X_3 belong to the line PQ.

14. Let us recall the statement according to which the circle l is invariant under the inversion with respect to the circle k if and only if $l = k$ or $l \perp k$.

    Since the point M belongs to the polar of the point A with respect to k we have $\angle MA^*A = 90^\circ$ where $A^* = \psi_l(A)$. Therefore $A^* \in l$ where l is the circle with the radius AM. Analogously $M^* \in l$. However from $A \in l$ we get $A^* \in l^*$; $A^* \in l$ yields $A \in l^*$ (the inversion is inverse to itself) hence $\psi_l(A^*) = A$). Similarly we get $M \in l^*$ and $M^* \in l^*$. Notice that the circles l and l^* have the four common points A, A^*, M, M^*, which is exactly two too much. Hence $l = l^*$ and according to the statement mentioned at the beginning we conclude $l = k$ or $l \perp k$. The case $l = k$ can be easily eliminated, because the circle l has the diameter AM, and AM can't be the diameter of k because A and M are conjugated to each other.

    Thus $l \perp k$, q.e.d.

15. Let $J = KL \cap MN$, $R = l \cap MN$, $X_\infty = l \cap AM$. Since MN is the polar of A from $J \in MN$ we get $\mathcal{H}(K,L;J,A)$. From $KLJA \overset{M}{\barwedge} PQRX_\infty$ we also have $\mathcal{H}(P,Q;R,X_\infty)$. This implies that R is the midpoint of PQ.

16. Let Q be the intersection point of the lines AP and BC. Let Q' be the point of BC such that the ray AQ' is isogonal to the ray AQ in the triangle ABC. This exactly means that $\angle Q'AC = \angle BAQ$ i $\angle BAQ' = \angle QAC$.

    For an arbitrary point X of the segment BC, the sine theorem applied to triangles BAX and XAC yields

$$\frac{BX}{XC} = \frac{BX}{AX}\frac{AX}{XC} = \frac{\sin\angle BAX}{\sin\angle ABX}\frac{\sin\angle ACX}{\sin\angle XAC} = \frac{\sin\angle ACX}{\sin\angle ABX}\frac{\sin\angle BAX}{\sin\angle XAC} = \frac{AB}{AC}\frac{\sin\angle BAX}{\sin\angle XAC}.$$

    Applying this to $X = Q$ and $X = Q'$ and multiplying together afterwards we get

$$\frac{BQ}{QC}\frac{BQ'}{Q'C} = \frac{AB}{AC}\frac{\sin\angle BAQ}{\sin\angle QAC}\frac{AB}{AC}\frac{\sin\angle BAQ'}{\sin\angle Q'AC} = \frac{AB^2}{AC^2}. \quad (9)$$

    Hence if we prove $BQ/QC = AB^2/AC^2$ we would immediately have $BQ'/Q'C = 1$, making Q' the midpoint of BC. Then the line AQ is isogonaly conjugated to the median, implying the required statement.

    Since P belongs to the polars of B and C, then the points B and C belong to the polar of the point P, and we conclude that the polar of P is precisely BC. Consider the intersection D of the line BC with the tangent to the circumcircle at A. Since the point D belongs to the polars of A and P, AP has to be the polar of D. Hence $\mathcal{H}(B,C;D,Q)$. Let us now calculate the ratio BD/DC. Since the triangles ABD and CAD are similar we have $BD/AD = AD/CD = AB/AC$. This implies $BD/CD = (BD/AD)(AD/CD) = AB^2/AC^2$. The relation $\mathcal{H}(B,C;D,Q)$ implies $BQ/QC = BD/DC = AB^2/AC^2$, which proves the statement.

17. The statement follows from the Brianchon's theorem applied to $APBMCN$.

18. Applying the Brianchon's theorem to the hexagon $AMBCPD$ we get that the line MP contains the intersection of AB and CD. Analogously, applying the Brianchon's theorem to $ABNCDQ$ we get that NQ contains the same point.

19. The Brokard's theorem claims that the polar of $F = AD \cap BC$ is the line $f = EO$. Since the polar of the point on the circle is equal to the tangent at that point we know that $K = a \cap d$, where a and d are polars of the points A and D. Thus $k = AD$. Since $F \in AD = k$, we have $K \in f$ as well. Analogously we can prove that $L \in f$, hence the points E, O, K, L all belong to f.

20. Let $G = AD \cap BC$. Let k be the circumcircle of $ABCD$. Denote by k_1 and k_2 respectively the circumcircles of $\triangle ADF$ and $\triangle BCF$. Notice that AD is the radical axis of the circles k and k_1; BC the radical axis of k and k_2; and FH the radical axis of k_1 and k_2. According to the famous theorem these three radical axes intersect at one point G. In other words we have shown that the points F, G, H are colinear.

    Without loss of generality assume that F is between G and H (alternatively, we could use the oriented angles). Using the inscribed quadrilaterals $ADFH$ and $BCFH$, we get $\angle DHF = \angle DAF = \angle DAC$ and $\angle FHC = \angle FBC = \angle DBC$, hence $\angle DHC = \angle DHF + \angle FHC = \angle DAC + \angle DBC = 2\angle DAC = \angle DOC$. Thus the points D, C, H, and O lie on a circle. Similarly we prove that the points A, B, H, O lie on a circle.

    Denote by k_3 and k_4 respectively the circles circumscribed about the quadrilaterals $ABHO$ and $DCHO$. Notice that the line AB is the radical axis of the circles k and k_3. Simlarly CD and OH, respectively, are those of the pairs of circles (k, k_2) and (k_3, k_4). Thus these lines have to intersect at one point, and that has to be E. This proves that the points O, H, and E are colinear.

    According to the Brocard's theorem we have $FH \perp OE$, which according to $FH = GH$ and $OE = HE$ in turn implies that $GH \perp HE$, q.e.d.